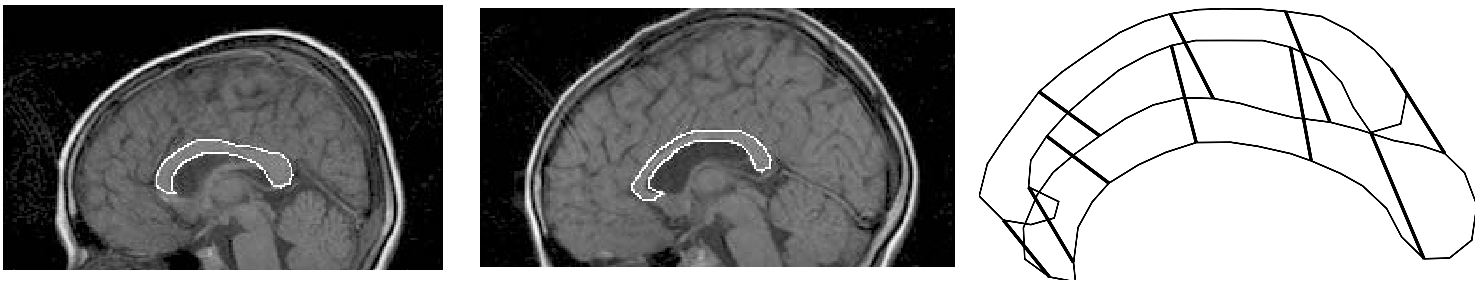
Non−rigid Curve Correspondence

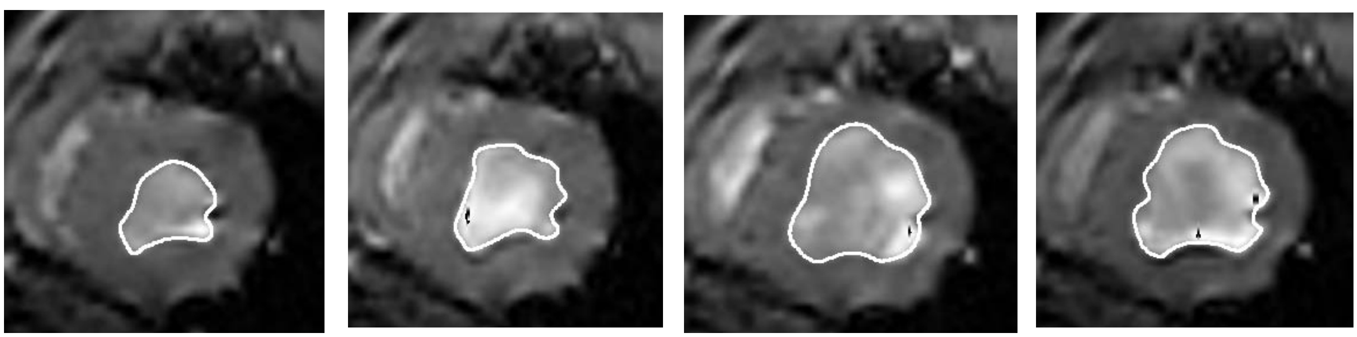
and

Its Application to Medical Imaging

Matching structures across subjects

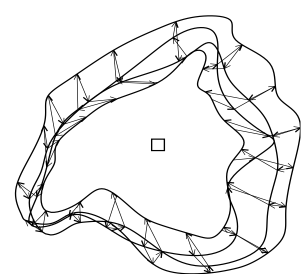


Motion

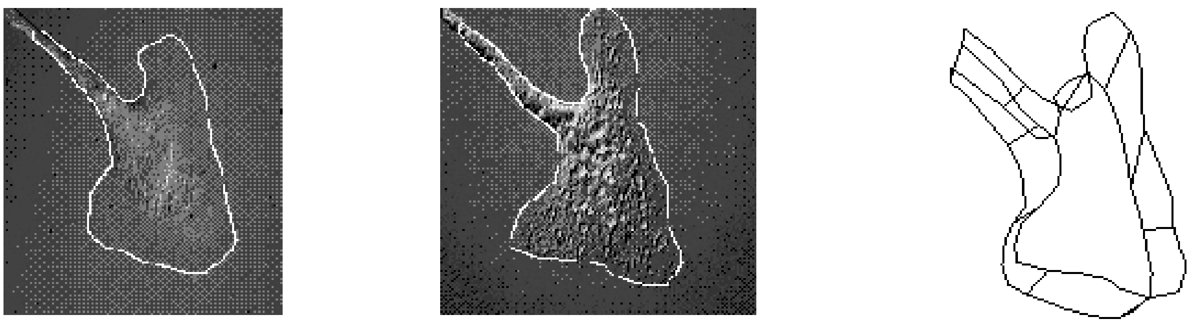


End Systole

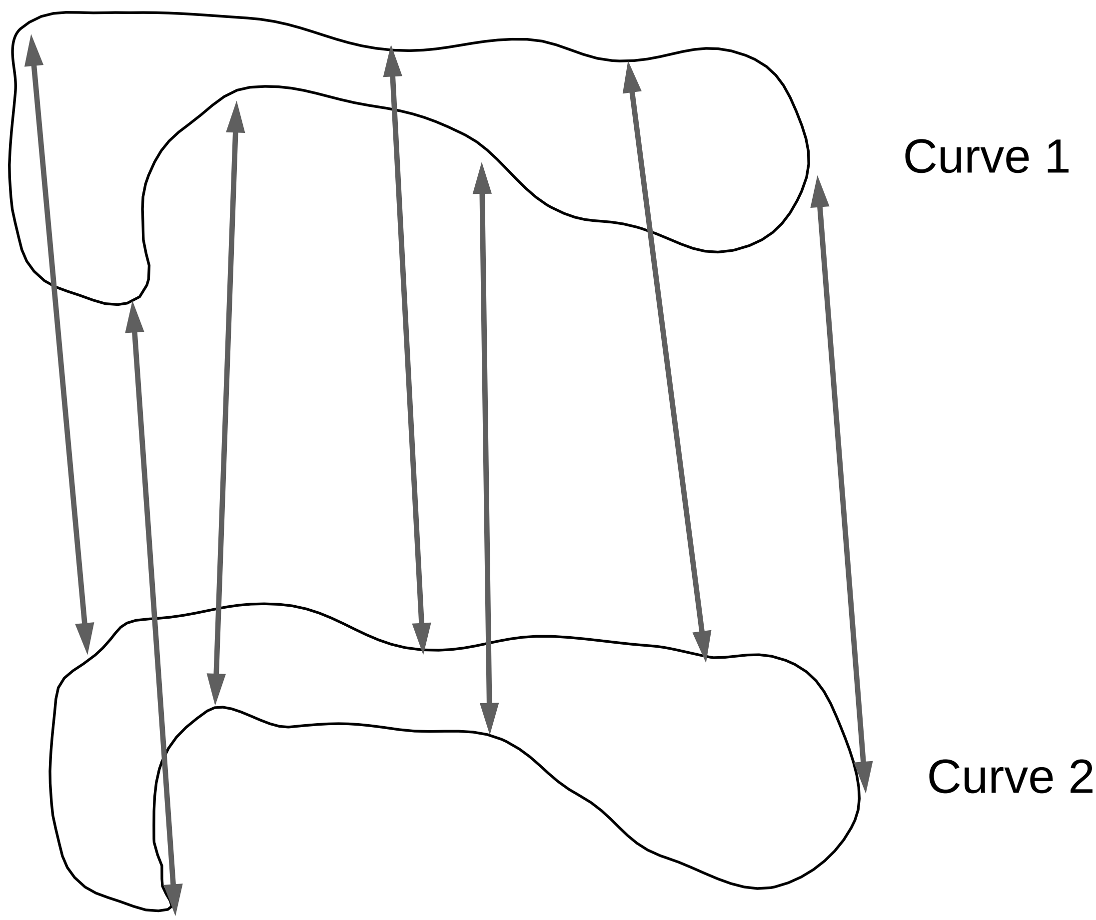
End Diasotle



Growth

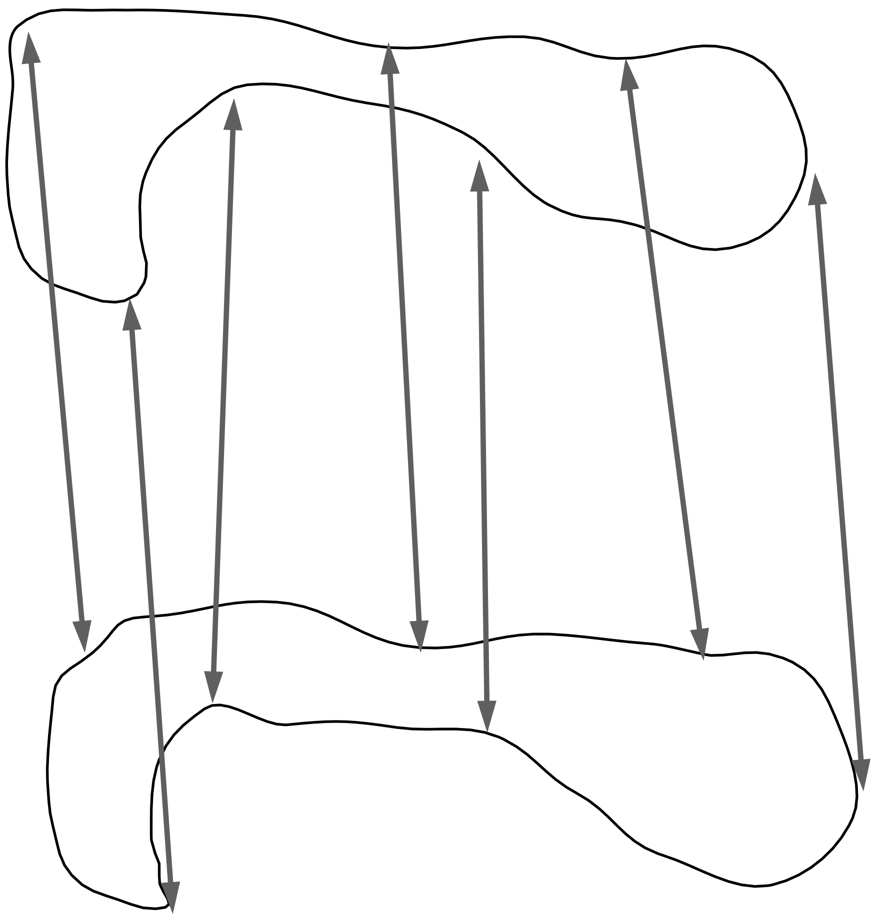


Given a pair of curves, find the point−wise correspondence that best aligns local shape



Why not compare curvatures?

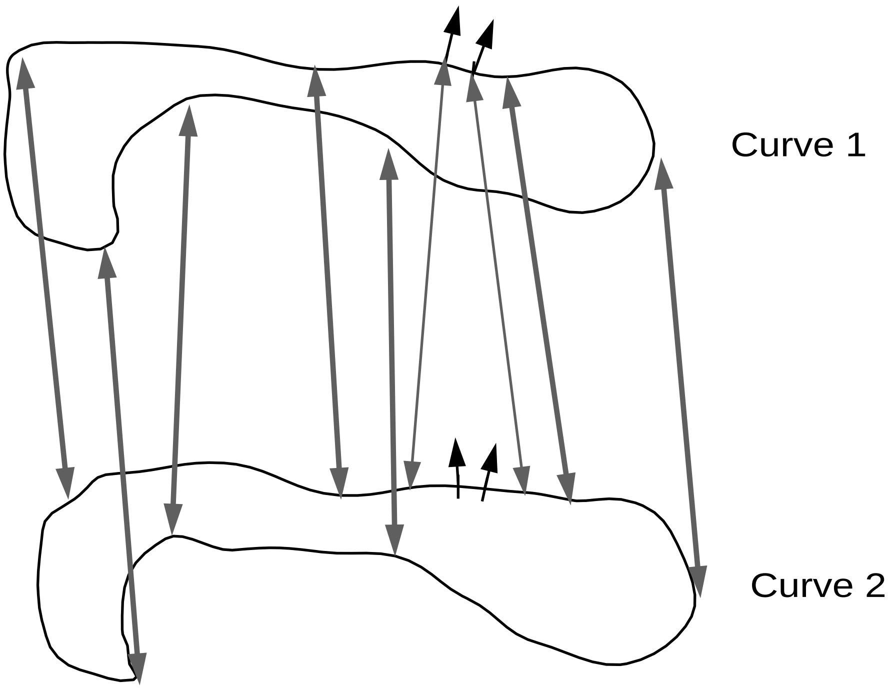
One approach.



$f: (0,L_1) \rightarrow (0,L_2) \quad s_2 = f(s_1), \quad \text{s.t. } f'(s_1) > 0.$

$$\text{min w.r.t. } f \quad J(f;C_1,C_2) = \int_{L_1} (k_2\,(f(s_2)) - k_1(s_1))^2 \, ds_1$$

Why not compare curvatures?



[1] Curvature is a rigid invariant −− but the correspondence may be non−rigid.

[2] The result depends on which curve we label 1 and which we label 2. The minimum changes according to the label !

$$J(f;C_1,C_2) = \int_{L_1} (k_2\,(f(s_2)) - k_1(s_1))^2 \; ds_1$$

$ds2 = f'(s1)\, ds1$

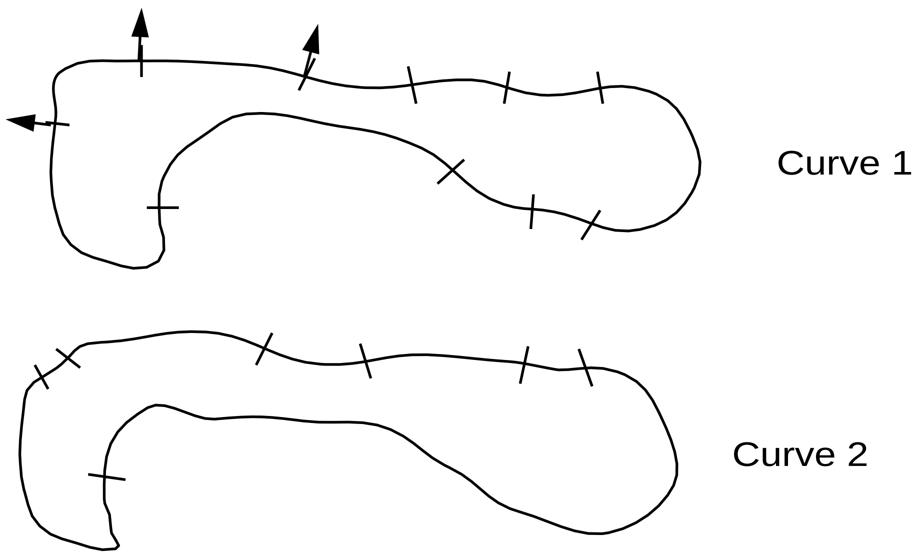
$ds_1 = 1/f'(s_1)\, ds_2$

$= f^{-1'}(s_2)\, ds_1$

$$= \int_{L_1} (k_2\,(s_2) - k_1(f^{-1}(s2)))^2 \; \underline{f^{-1'}(s2)} \; ds_2$$

$$\neq \int_{L_1} (k_2\,(s_2) - k_1(f^{-1}(s2)))^2 \; ds_2$$

String Edit Distances



$$S1 = (\Delta\theta_1, \Delta\theta_2 \quad \ldots\ldots \quad)$$

$$S2 = (\Delta\bar{\theta}_1, \Delta\bar{\theta}_2 \quad \ldots\ldots \quad)$$

Replacement rules: $\Delta\theta \rightarrow \Delta\bar{\theta}$ Cost: c

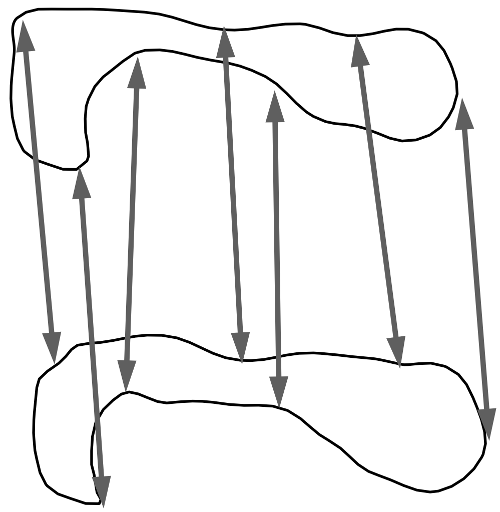
$\Delta\theta \rightarrow \{\}$ Cost a

$\{\} \rightarrow \Delta\theta$ Cost

D(C1, C2) = minimum string edit cost (distance) Levenstein dist.

The optimal edits give a correspondence between segments.

What do we want?



A theory of point−wise,

non−rigid correspondence:

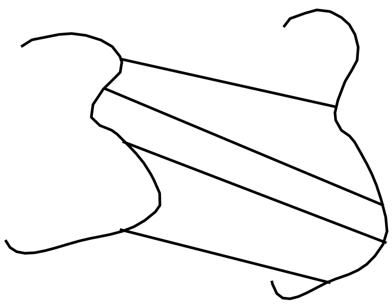
[a] which allows for 1−1 correspondences,

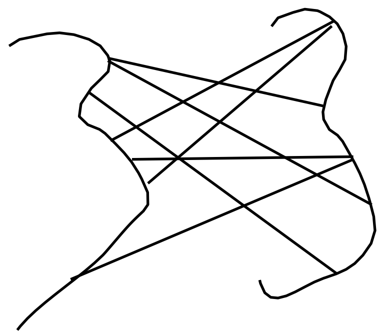
[b] which is symmetric with respect to curve labels,

[c] which uses shape properties of the curves.

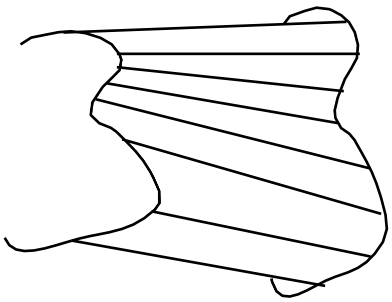
Outline

Define a correspondence between curves C_1 and C_2 .

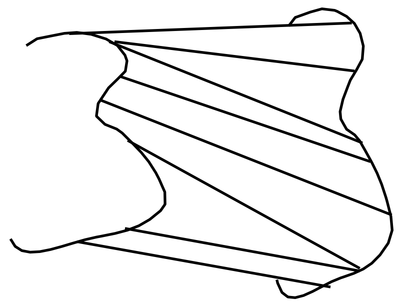




Define the "feasible" subset of correspondences.



1−1



non 1−1

Outline

Evaluate how well a correspondece aligns the local shape of curves?

Objective function for non−rigid shape comparison.

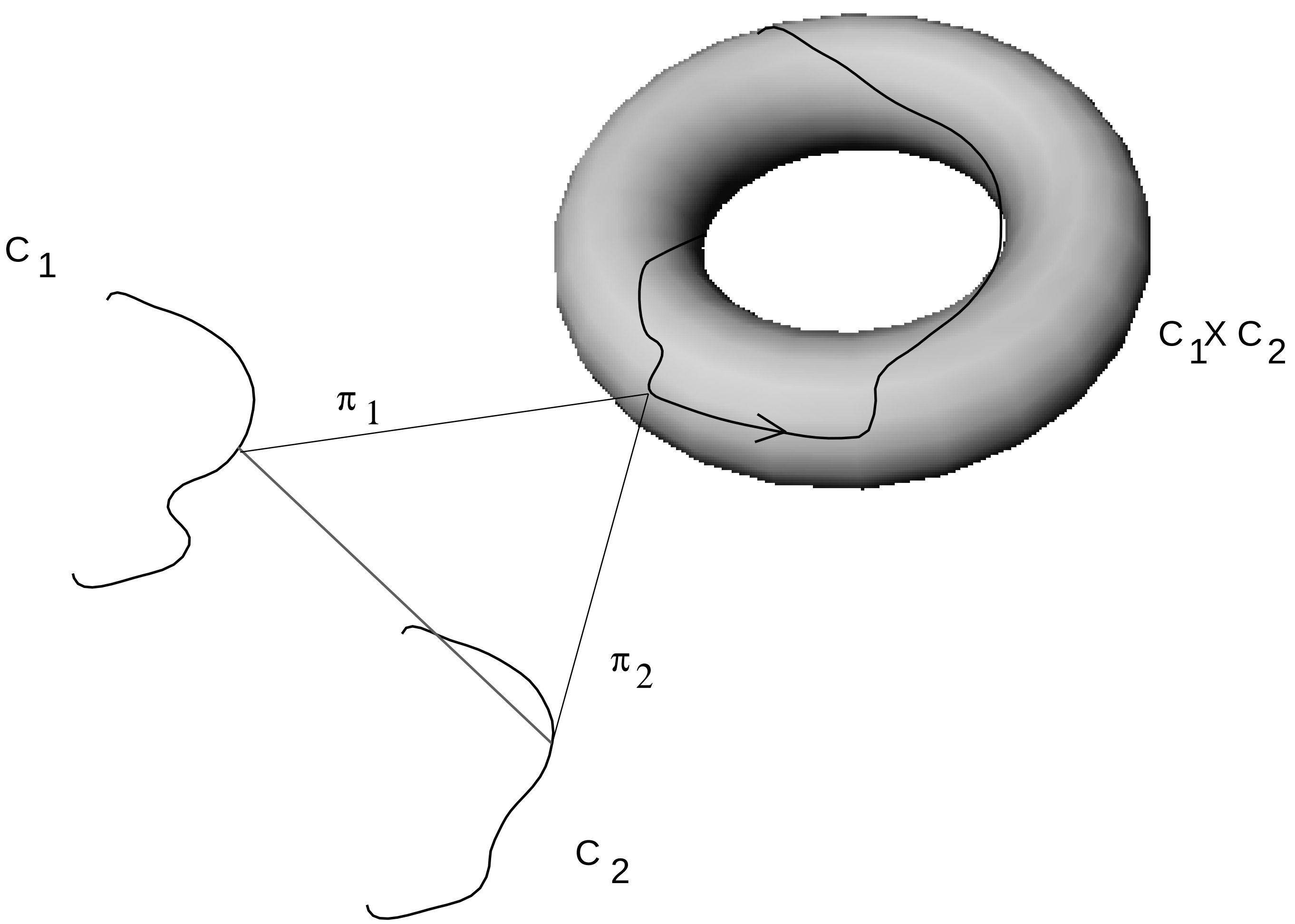
Numerical procedure for searching through the set of bi−morphisms to find the "best" one w.r.t. the objective function

Correspondence

<u>Definition:</u> A correspondence between C_1 and C_2 is a subset of $C_1 X C_2$ whose projections on C_1 and C_2 are onto.

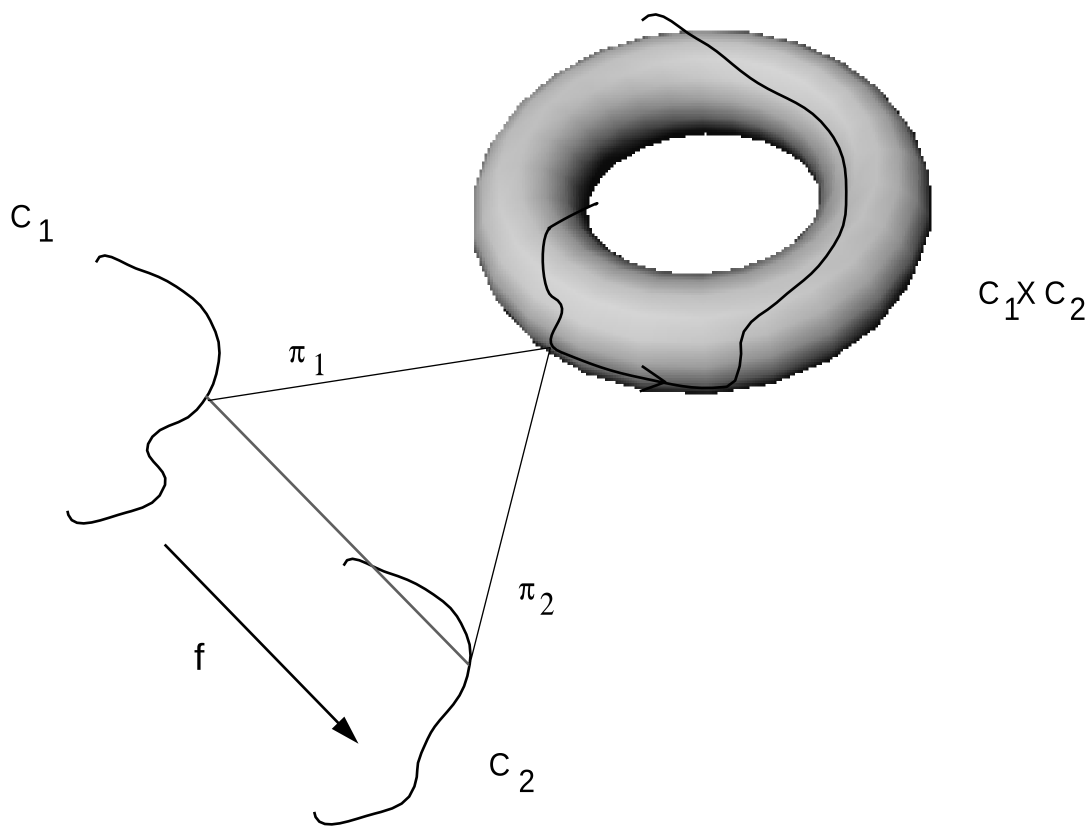
$$\Phi = \{ (u,v),\ u \,\varepsilon\, C_1 \quad v \,\varepsilon\, C_2 \},$$

$$\pi_1(\Phi) = C_1 \qquad \pi_2(\Phi) = C_2 \ .$$



Correspondence

Definition: By a diffeomorphic correspondence we mean a correspondence which is the graph of a diffeomorphism from $C_1 \to C_2$ (or $C_2 \to C_1$).



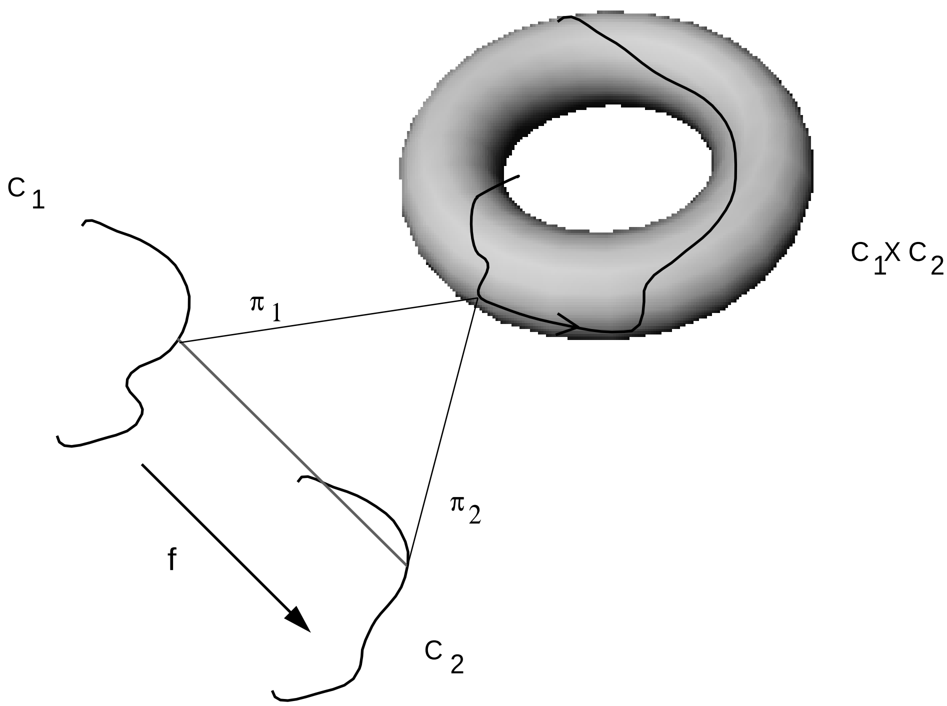
Note: [1] These correspondences are one−to−one,

[2] The correspondence may or may not be in accordance with the orientations of the two curves.

What is its structure in the product space?

Correspondence

<u>Theorem:</u> A correspondence is diffeomorphic iff it is a regular curve in C_1 X C_2 whose projections on C_1 and C_2 are onto and diffeomorphisms.



<u>Proof:</u> [1] Let $C_1(t)$:I−>R be a regular parameterization of C_1. Then, consider F:I−>C1 X C2 given by $F(t) = (C_1(t), f(C_1(t)))$. It is a regular parameterization of the correspondence.

The projections of any correspondence are onto.

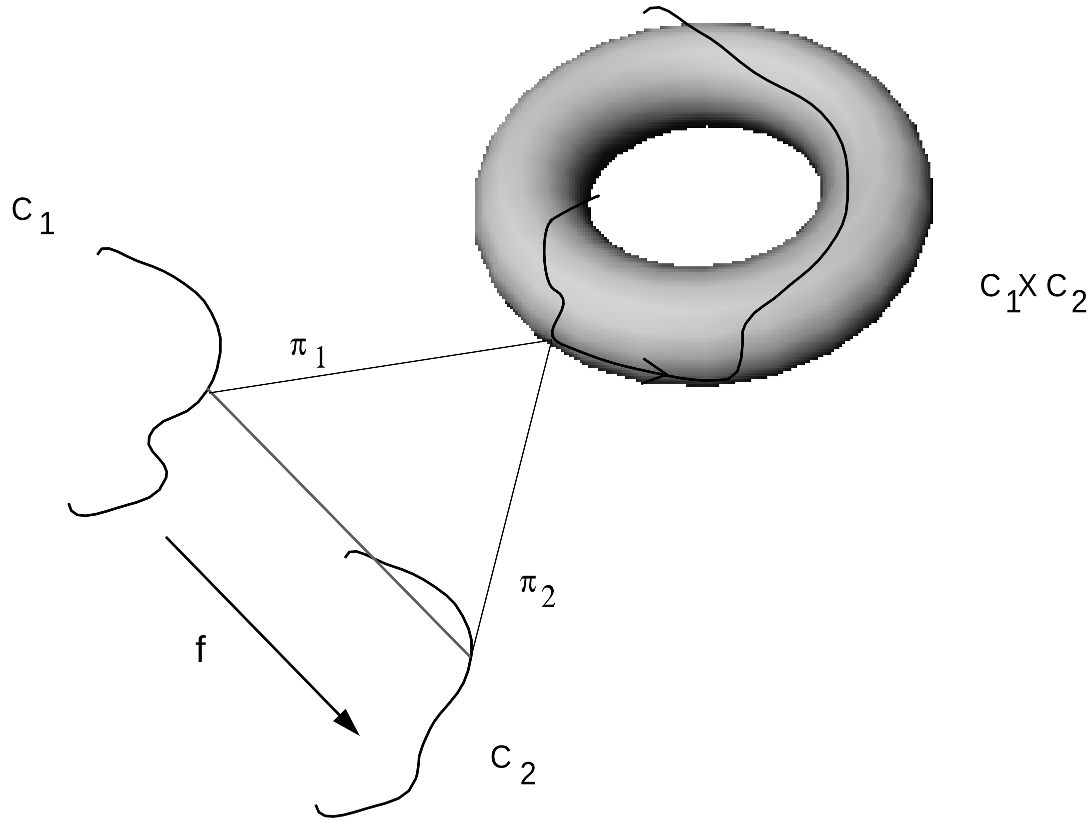
Note that $\pi 1((C1(t_1), f(C1(t_1))) = \pi 1((C1(t_2), f(C1(t_2)))$

$\Leftrightarrow$ $C_1(t_1) = C_1(t_2)$.

Similarly $\pi 2((C1(t1), f(C1(t1))) = \pi 2((C1(t2), f(C1(t2)))$

$\Leftrightarrow$ $f(C_1(t_1)) = f(C_1(t_2))$.

Correspondence



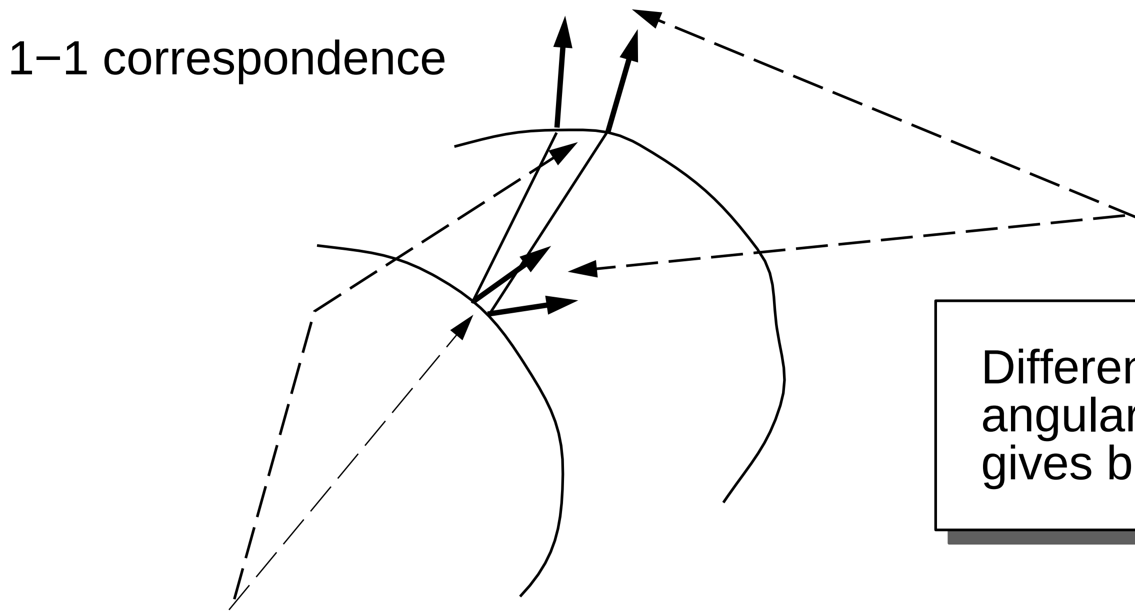
<u>Proof:</u> [2] Let F be a regular curve in C_1 X C_2 whose projections on C_1 and C_2 are onto and diffeomorphisms.

Consider f:C_1 −> C_2 given by

$$f = \pi_2 \text{ o } \pi_1^{-1}.$$

It is onto and is also a diffeomorphism ('cause it is a composition of diffeomorphisms).

Non−rigid Shape Comparison

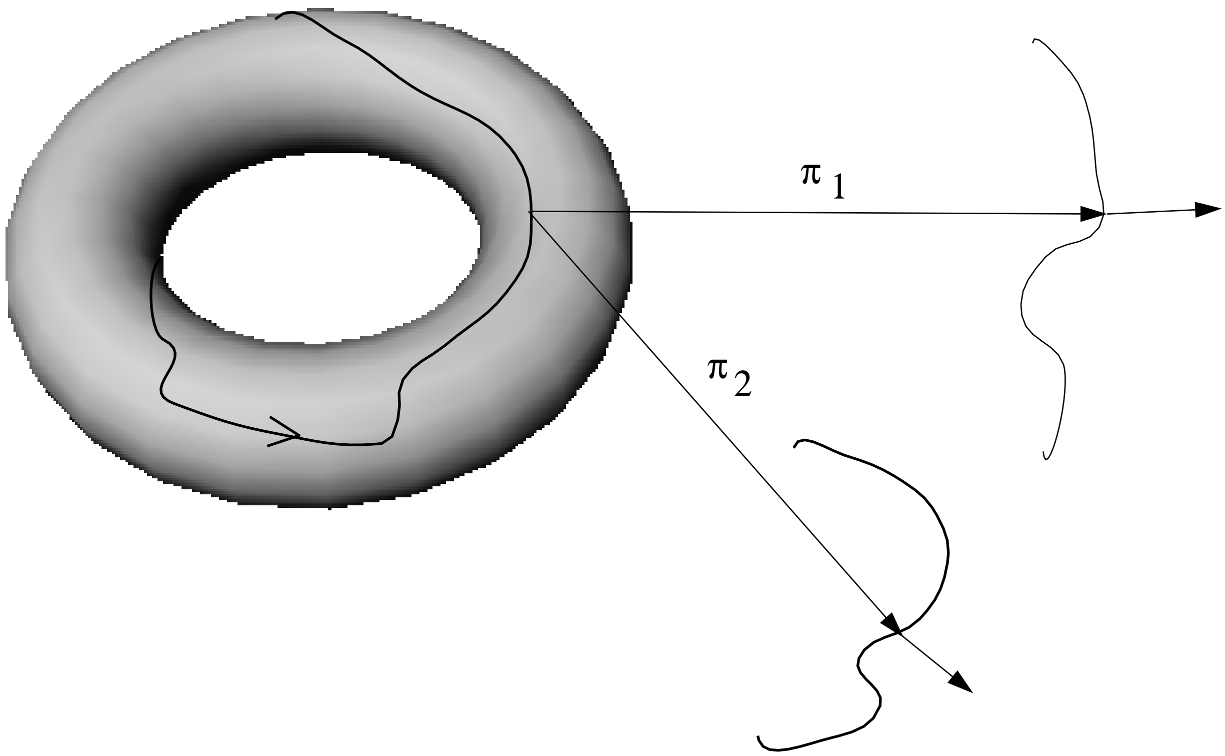


Difference in the angular change gives bending

Difference (or ratio) of arc−length gives stretching

How do you express the difference symmetrically?

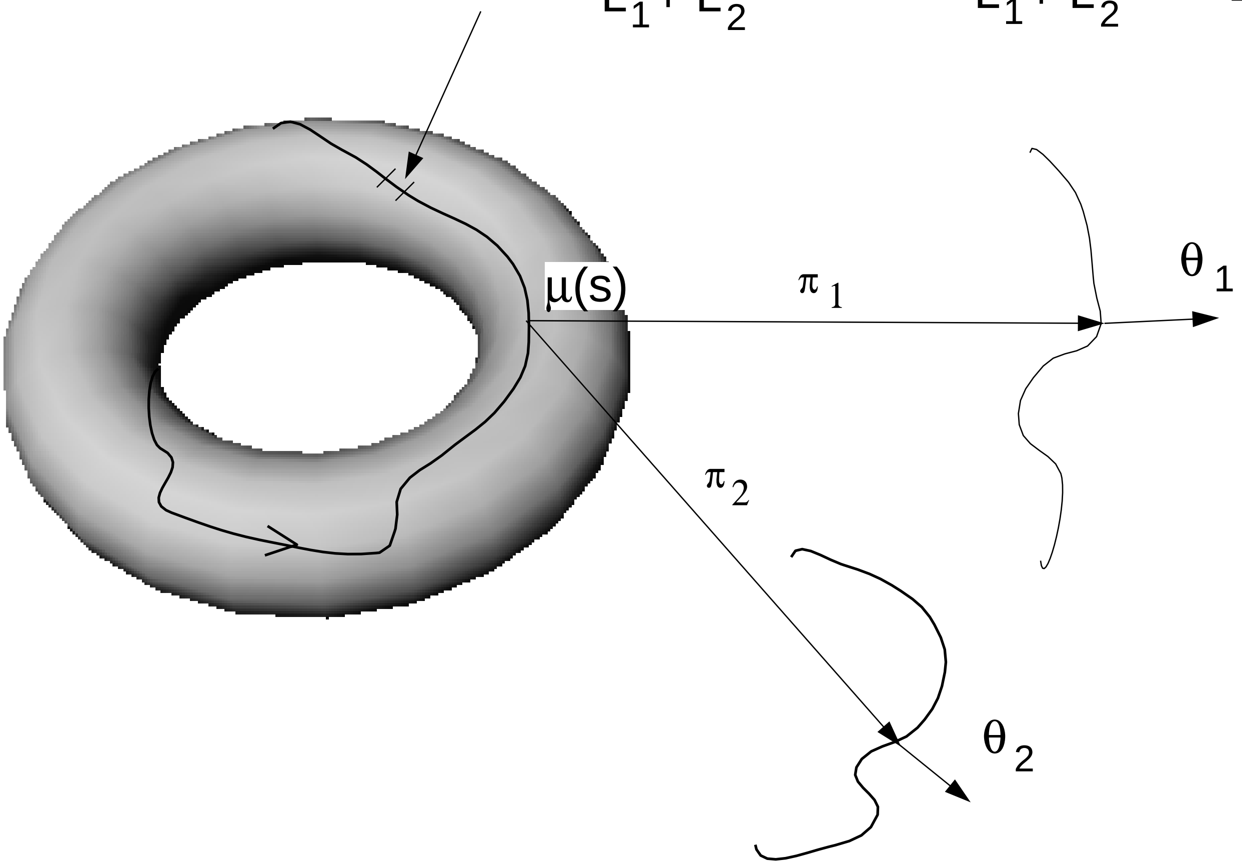
Do all comparisons from the bi−morphism !!



This gives a symmetric formulation as well as allows for comparison via a non 1−1 bi−morphism.

Geometry

$$ds^2 = \frac{L_1}{L_1 + L_2}\, ds_1^2 + \frac{L_2}{L_1 + L_2}\, ds_2^2$$



$$J = \int \left(\frac{d\theta_1(\pi_1(\mu(s)))}{ds} - \frac{d\theta_2(\pi_2(\mu(s)))}{ds} \right)^2 ds$$

$$+ \lambda \int \left(\frac{d \,\|\, \pi_1'(\mu(s)) \,\|}{ds} \right)^2 + \left(\frac{d \,\|\, \pi_2'(\mu(s)) \,\|}{ds} \right)^2 ds$$

Numerical Procedure

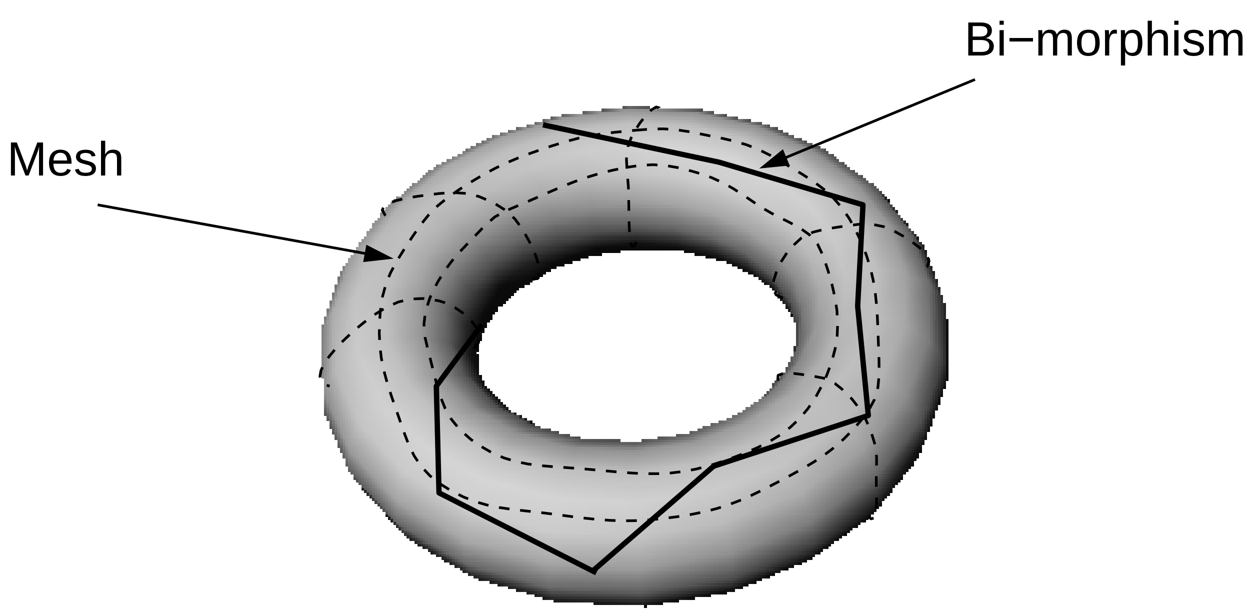
Calculate the bi−morphism that minimizes J

[1] Adaptively filter the curve to reduce noise,

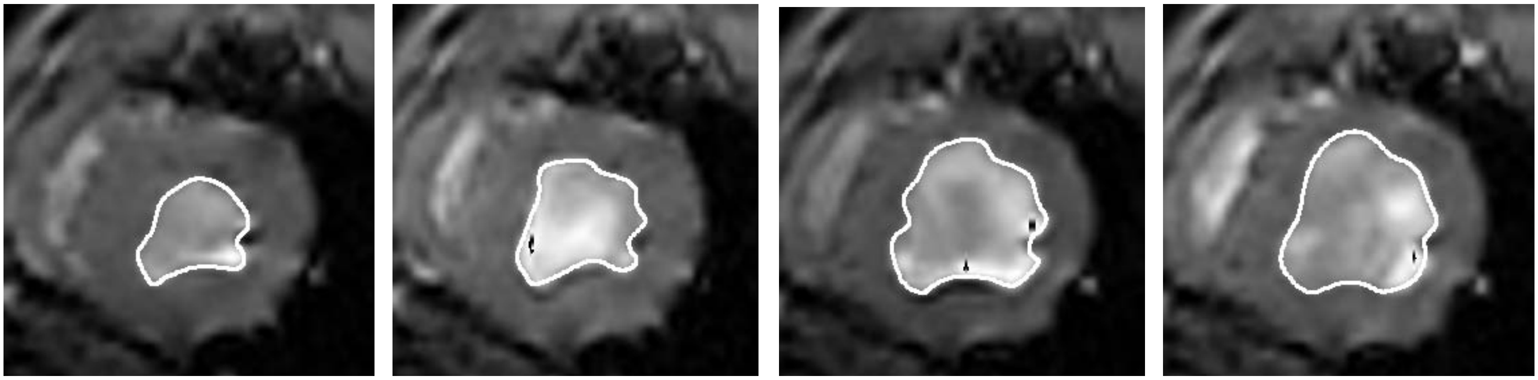
[2] Approximate the bi−morphism with a finite element,

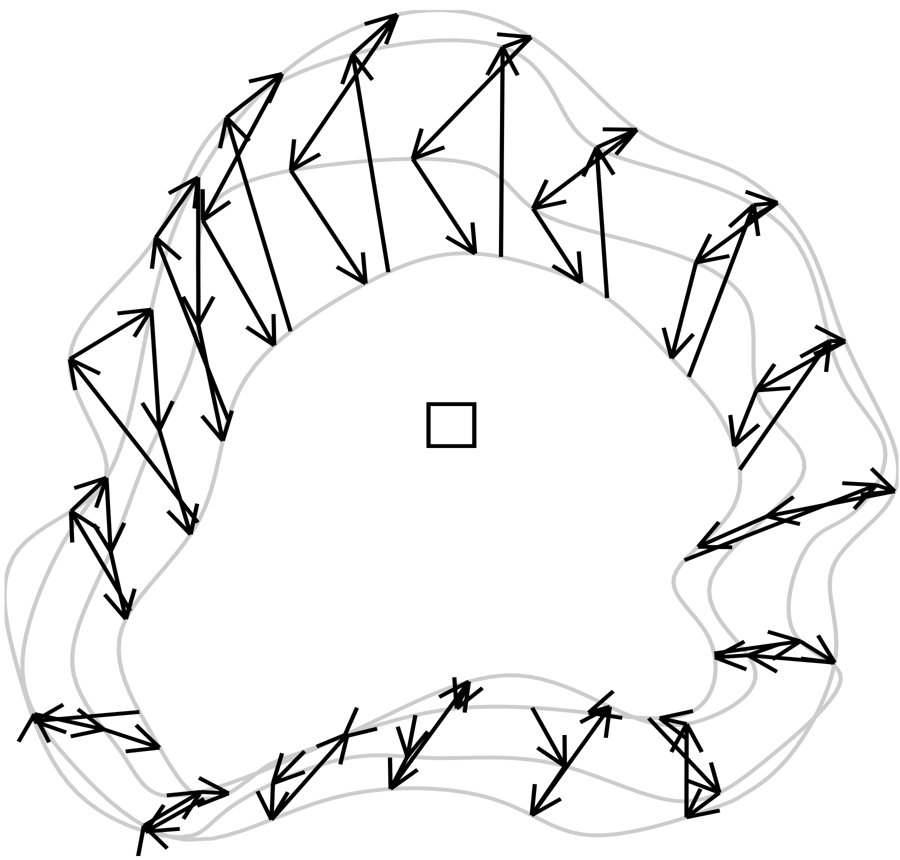
[3] Use dynamic programming on a mesh for initialization,

[4] Continuous descent for fine minimization

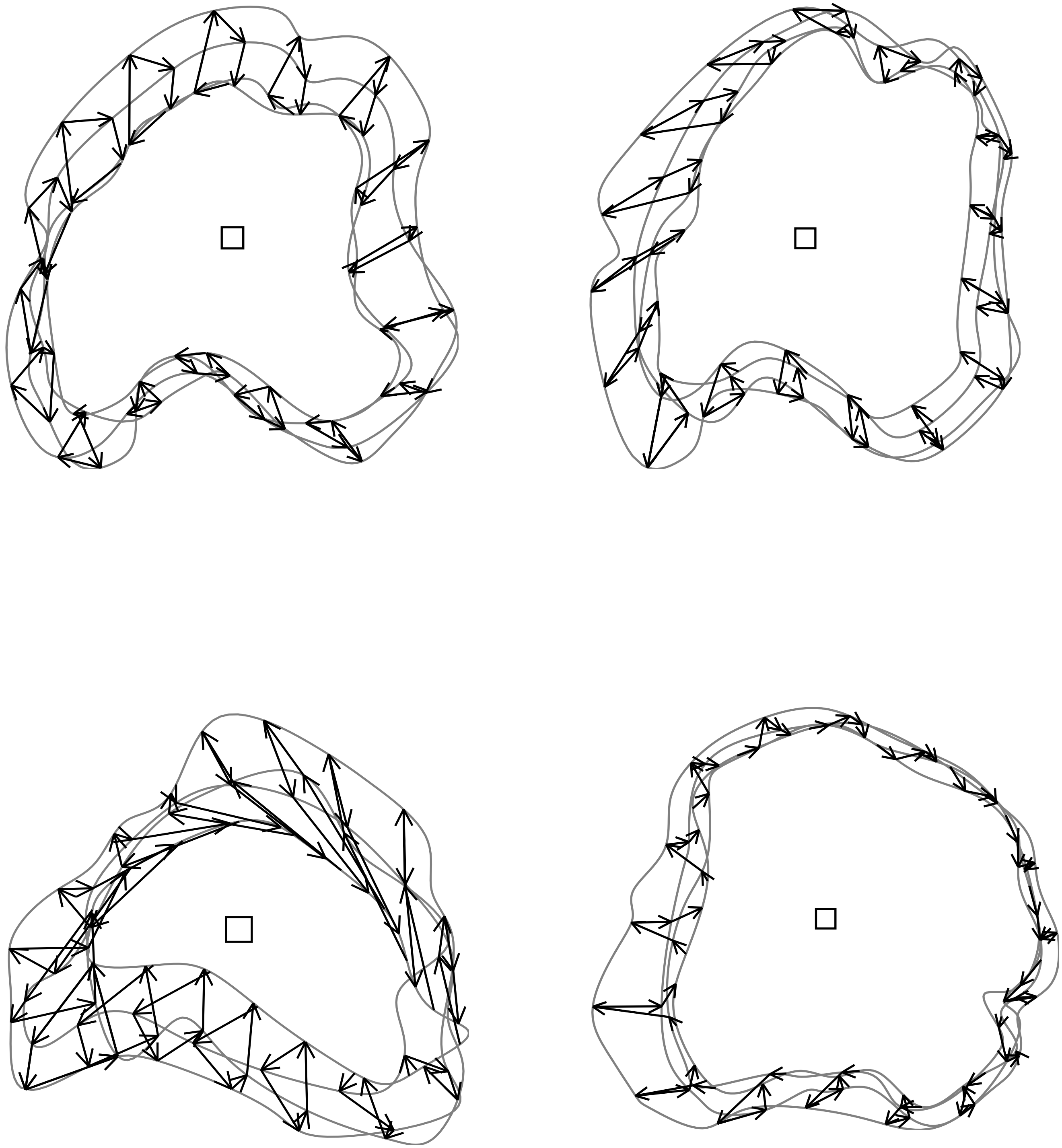


Experimental Results

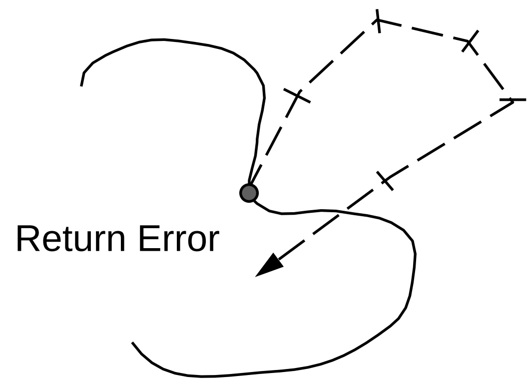


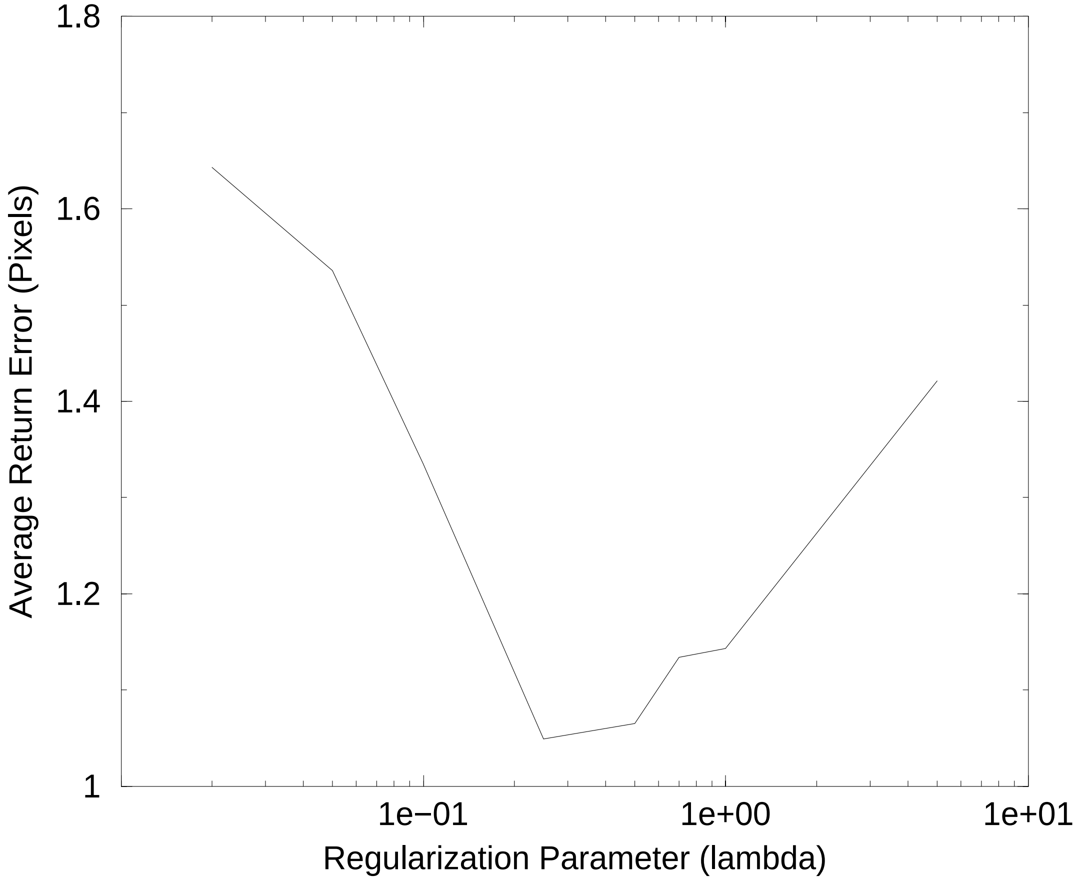


Experimental Results



Experimental Results

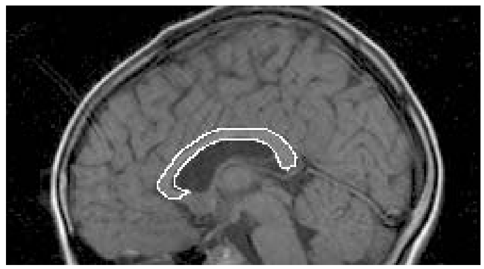
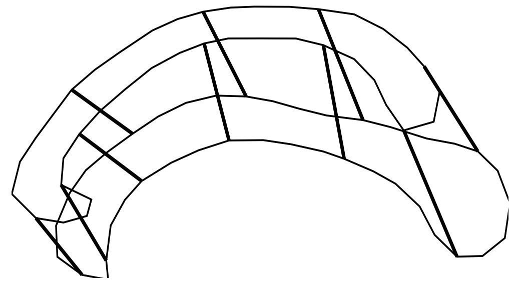


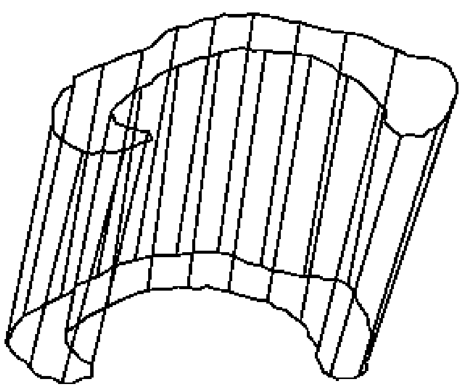
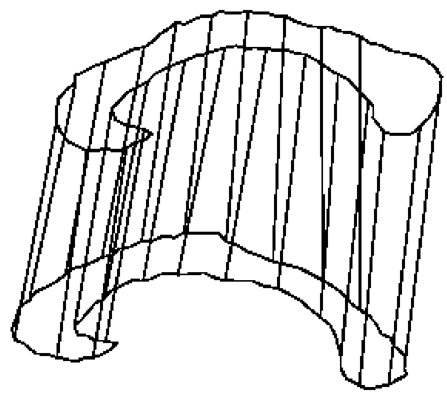


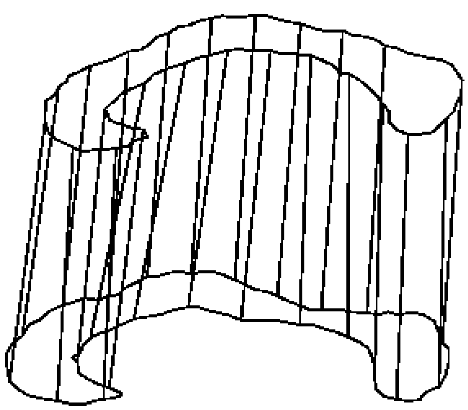
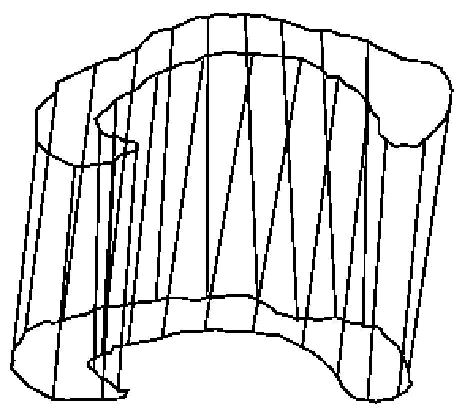
Return Error (Pixels)

Regularization Parameter (λ)

Experimental Results

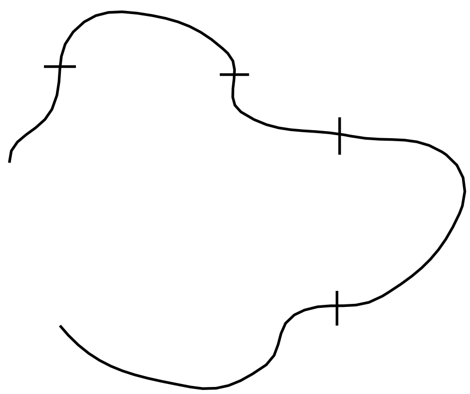



Non−rigid Shape Invariants

When can we find a diffeomorphic correspondence which gives

$$\frac{d\theta_1(\pi_1(\mu(s)))}{ds} = \frac{d\theta_2(\pi_2(\mu(s)))}{ds} \quad ? \qquad (*)$$



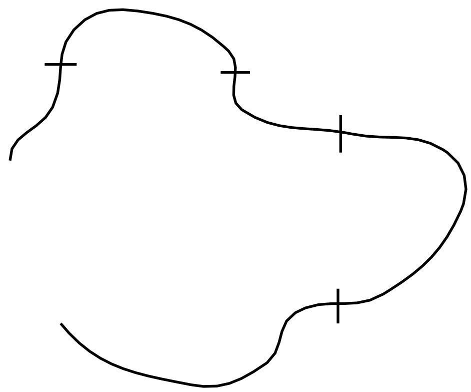
$(\Delta\theta_1, \Delta\theta_2 \quad ..... \quad)$

Angular deviation string

Theorem: The equality (*) is possible iff the two curves have identical angular deviation strings (modulo circular shift).

Note: We need some technical conditions for this result.

Non−rigid Shape Invariants



$(\Delta\theta_1, \Delta\theta_2 \; ..... \;)$

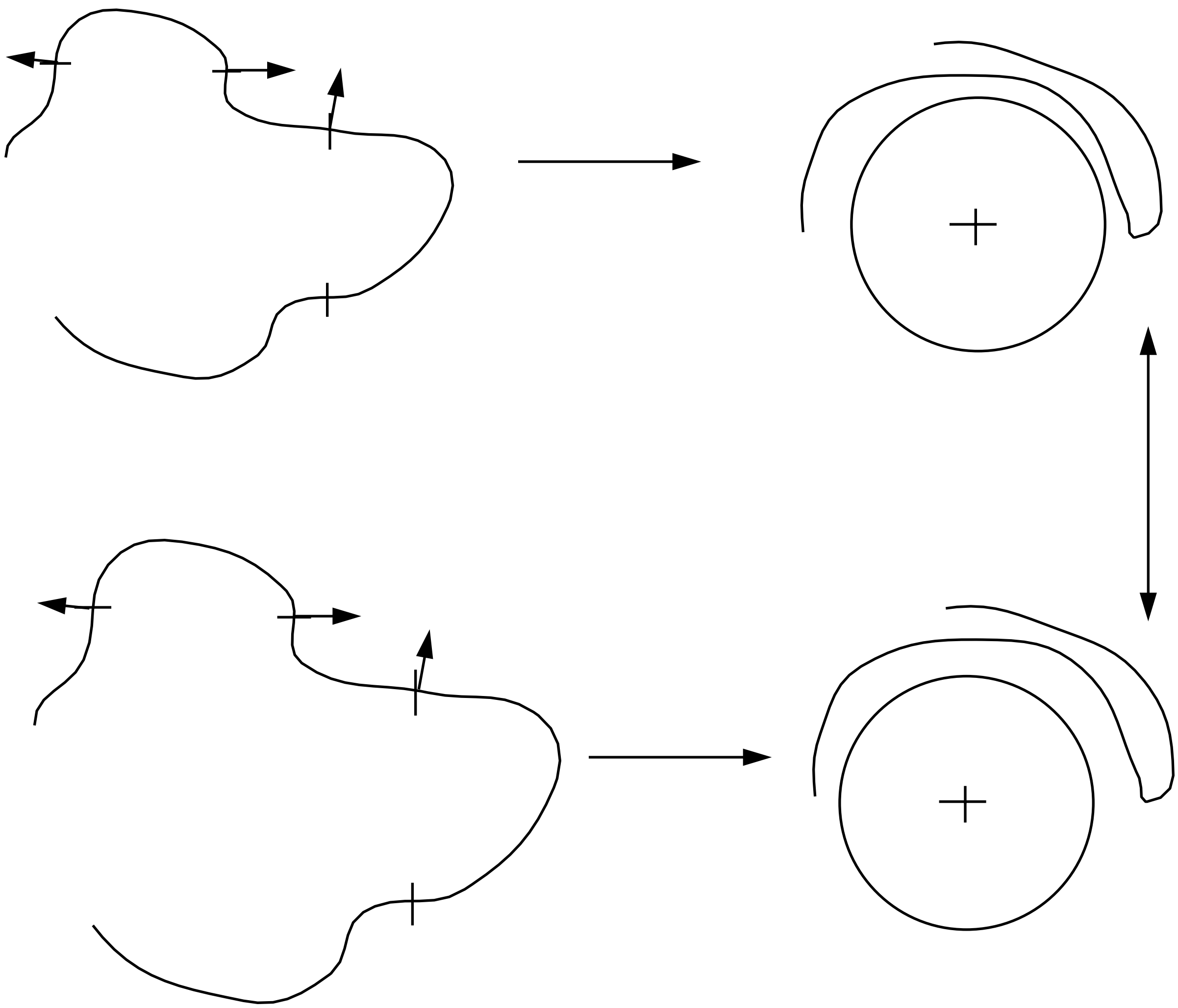
Angular deviation string

<u>Theorem:</u> The equality (*) is possible iff the two curves have identical angular deviation strings (modulo circular shift).

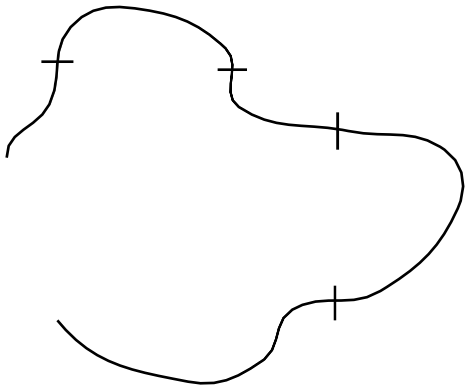
Outline of proof:

[1] Integrate (*) w.r.t. s.

[2]



Non−rigid Shape Invariants



$(\Delta\theta_1, \Delta\theta_2 \; ..... \;)$

Angular deviation string

Theorem: The angular deviation string partitions the set of curves into equivalence classes.

The string edit distance is a distance between the equivalence classes of curves.