Affine Connections in Plain English

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Recall:

A tangent vector is a directional derivative operator at a point.

The set of all directional derivative operators at a point is a vector-space and is called the tangent space at the point.
The notation is a little funky:

We will denote tangent vectors as $\nu_p$ etc.

The directional derivative of a function is denoted $\nu [f]$

Given a parametrization, the co-ordinate curves form a basis of the tangent space. This is expressed as

$$\nu_p = \sum_i a_i \left( \frac{d}{dx_i} \right)_p$$

$$\nu_p [f] = \sum_i a_i \left( \frac{d}{dx_i} \right)_p [f]$$

The tangent space is denoted $T_p M$, where $M$ is the manifold.
The set of all tangent spaces of a manifold is the tangent bundle of the manifold, denoted TM. The tangent bundle is itself a manifold:

If \( \{ (U_\alpha, \phi_\alpha) \} \) is an atlas for M, then

\[
\{(U_\alpha \times \mathbb{R}^n, (\phi_\alpha, \gamma_\alpha))\}
\]

is an atlas for TM:

where, \( \gamma_\alpha \) takes \( \mathbb{R}^n \) into the tangent space at \( p \).
A vector field on a manifold $M$ is a correspondence (function) which associates with each point $p$ of $M$ a vector $x(p)$ of $T_p M$.

The vector field is $x$, its value at a point is $x(p)$.

A vector field $x$ is a map from $M$ to $TM$. $x : M \rightarrow TM$.

The vector field is differentiable if the map $x : M \rightarrow TM$ is differentiable.

**Theorem:** Let a vector field $v$ be expressed in local co-ordinates as $v(p) = \sum_i a_i(p) \left( \frac{d}{dx_i} \right)_p$, then $v$ is differentiable if and only if $a_i(p)$ are differentiable functions on the manifold.
Define: Let \( v \) be a vector field on manifold \( M \). An integral curve of \( v \) passing through point \( P \) of \( M \) is a curve \( t \rightarrow C(t) \) such that

\[
[1] \quad C(0) = P, \quad \text{and} \quad [2] \quad \frac{d}{dt} C(t) = v(C(t)).
\]
After Riemann, the next biggest conceptual leap was by Levi-Civita (1917) who formalized the notion of parallel transport, connection and hence that of a geodesic.

A clear definition of these quantities was not available even though mathematicians as far back as Euler understood what a geodesic was.

Curve of least “intrinsic curvature”
Geodesic

Curve of least "intrinsic curvature"

Move this vector parallel to itself through ambient space

Difference in the two tangent vectors

The component in the tangent plane tells us how much the curve appears to swing in the surface.

This is the notion of "intrinsic curvature"
Do this construction differentially

A geodesic is a curve for which the projection of the derivative of the tangent vector onto the tangent plane is zero at all points of the curve.

**Theorem**: Locally, a geodesic minimizes arc-length.

**Note**: The derivative of a tangent vector along a curve still requires the ambient space.
Projection of derivative of a tangent vector

(Surface in 3-D)

Vector field \( \mathbf{v} \)

Choose a direction \( \mathbf{w} \) also in the tangent space at \( P \)
Projection of derivative of a tangent vector

(Surface in 3-D)

[3]

Take the derivative of $v$ along $w$ at $P$

\[
\frac{dv}{dw} \bigg|_P
\]

[4]

Project it on the tangent plane at $P$

\[
\Pi_p \frac{dv}{dw} \bigg|_P
\]

$D (v) = \Pi_p \frac{dv}{dw} \bigg|_P$

$D (v)$ is called the **covariant derivative** of $v$ along $w$ at $P$. 

$w, P$
**Covariant Derivative**

By using properties of ordinary derivatives we can show that

\[
[1] \ D_{w,\mathcal{P}} (ax + by) = a \ D_{w,\mathcal{P}}(x) + b \ D_{w,\mathcal{P}}(y),
\]
\[
[2] \ D_{av+bu,\mathcal{P}}(x) = a \ D_{v,\mathcal{P}}(x) + b \ D_{u,\mathcal{P}}(x)
\]

**Parallel Transport**

**Defn:** A vector is parallely transported along a curve if its covariant derivative is zero.

A geodesic parallely transports its tangent vector.
Let \( w \) be another vector field and define \( \nabla \) to be the operator that gives the covariant derivative of \( v \) wrt \( w \) at every point on the surface:

\[
\nabla : \text{TS} \times \text{TS} \rightarrow \text{TS} \quad \text{given by} \quad \nabla (v) \text{ at } P = D_{w,p} (v)
\]

\( \nabla \) is called the affine connection. It tells us how the tangent space at every point is “connected” to the tangent spaces around the point.

For a surface it is derived via ambient space.
Affine Connection

The laws of ordinary derivative give us the following properties for the affine connection

[1] $\nabla_{fV+gW}(x) = f\nabla_V(x) + g\nabla_W(x)$

[2] $\nabla_V (x+y) = \nabla_V (x) + \nabla_V (y)$

[3] $\nabla_V (fx) = f\nabla_V (x) + v[f] x$

How do we derive the affine connection for a manifold??

How are these “connected”? i.e. how should we define a covariant derivative (a “projected” derivative)?
Affine Connection on a Manifold

The definition of a manifold does not tell us anything about how tangent spaces should be connected.

That we are free to “connect” them in any way.

Choosing a “connection” is equivalent to choosing an affine connection operator.

Choose any $\nabla : TM \times TM \rightarrow TM$ as long as it satisfies the formal properties [1] - [3]

\[
[1] \quad \nabla_{f\mathbf{v} + g\mathbf{w}} (x) = f \nabla_{\mathbf{v}} (x) + g \nabla_{\mathbf{w}} (x)
\]

\[
[2] \quad \nabla_{\mathbf{v}} (x+y) = \nabla_{\mathbf{v}} (x) + \nabla_{\mathbf{v}} (y)
\]

\[
[3] \quad \nabla_{\mathbf{v}} (f \mathbf{x}) = f \nabla_{\mathbf{v}} (\mathbf{x}) + \nabla[f] \mathbf{x}
\]
Affine Connection on a Manifold

Theorem:

Let \( x(p) = \sum_i a_i(p) \left( \frac{d}{dx_i} \right)_p \) and \( y(p) = \sum_i b_i(p) \left( \frac{d}{dx_i} \right)_p \)

Then,

\[
\nabla_y (x) = \sum_k \left( \sum_{i,j} a_i(p)b_j(p) \Gamma^k_{ij} \right)_p \left( \frac{d}{dx_k} \right)_p + x[b(p)] \left( \frac{d}{dx_k} \right)_p
\]

Christoffel symbols \((n^3)\). Choose any functions. Choosing these corresponds to choosing the kth component of the co-variant derivative of the ith basis w.r.t. jth basis.
Affine Connection on a Manifold

Is there a meaningful and generally useful affine connection?

Surface in 3-D: Affine connection from ambient space
Manifolds: Affine connection from Riemannian Metric
**Riemannian Metric**

**Defn:** A Riemannian Metric on a manifold $M$ is an inner product $\langle \ , \ \rangle_p$ defined for the tangent space at every point $p$ of the Manifold.

Inner product $\langle \ , \ \rangle_p$ is bilinear, symmetric and positive definite.
Let \( x(p) = \sum_{i} a_{i}(p) \left( \frac{d}{dx_{i}} \right)_{p} \) and \( y(p) = \sum_{i} b_{i}(p) \left( \frac{d}{dx_{i}} \right)_{p} \).

Then, \( \langle x(p), y(p) \rangle = \sum_{i,j} g_{ij}(p) a_{i}(p) b_{j}(p) \).

These four functions on the manifold define the Riemannian Metric.
Is there an affine connection such that the “orthogonal frame” of the Riemannian Metric is parallel transported along itself?

**Theorem (Levi-Civita):** Yes!

**Proof:**

\[
\Gamma^k_{ij} = \frac{1}{2} \sum_m \left\{ \frac{d}{dx_i} g_{jm} + \frac{d}{dx_j} g_{mi} - \frac{d}{dx_m} g_{ij} \right\} \left( G^{-1}\right)_{mk}
\]

where \( G = \text{matrix } g_{ij} \)
Parallel Transport with a Riemannian Connection

(Intuitive)

Normalize unit ellipse to unit circle by scaling axes

Two vectors are parallel if they have the same co-ordinates after normalization.
Covariant Derivative

(Intuitive)

Normalize unit ellipse to unit circle by scaling axes

\[
\frac{d\text{ change}}{dt} \quad \text{is the co-variant derivative}
\]
Geodesic
(Intuitive)

Tangent vector is transported parallel to itself.

Theorem: A geodesic minimizes local Riemannian arc-length
**Theorem: (Intuitive)**

1. For every tangent vector in some open ball of size (metric) $e > 0$, there is a corresponding unique geodesic in the manifold.

2. Let $C_v(t)$ be the geodesic whose initial vector is $v$. Then the map $\exp(v) = C_v(1)$ takes $v$ to point in $M$.

$\exp(v)$ is a diffeomorphism from the open ball in the tangent space to an open set of the manifold.
The Exponential Map

The exponential map is incredibly important in understanding the “curvature” of the space because it lets us define a “sphere” on the manifold that has the same dimension as the manifold.

\[
\frac{\text{Area}}{\text{diam}^2} = \pi
\]

\[
\frac{\text{Area}}{\text{diam}^2} < \pi
\]

\[
\frac{\text{Area}}{\text{diam}^2} > \pi
\]
The ideas of tangent spaces, affine connections and Riemannian metric allow us to do geometry (use geometric reasoning) in many problems.

\[\text{e.g. Calculus of variations.}\]

It is possible to geometry under a wide variety of metrics (distance). The metric can often be tailored to the problem

\[\text{e.g. unbiased snakes}\]

More general geometries are also possible by suitable definitions of affine connection.

\[\text{e.g. affine differential geometry}\]

You can get additional mileage by adding more structure to this

\[\text{e.g. Lie groups}\]