Manifolds in Plain English

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References:
[1] Riemannian Geometry
    do Carmo
    Chris J Isham
[3] An Introduction to Differentiable Manifolds and
    Riemannian Geometry
    William Boothby
    Spivak
Diffeomorphisms

Maps (functions) from subsets of $\mathbb{R}^n$ to subsets of $\mathbb{R}^m$ can be differentiated. (Why?)

Derivative of $f: U \to W$ is denoted $df$

Defn: A function $f: U \to W$ is a diffeomorphism if it is a homeomorphism and if it is differentiable in both directions (i.e. $df$ and $d(f^{-1})$ exist).

Problem: Construct a $C^\infty$ homeomorphism $f: \mathbb{R} \to \mathbb{R}$ that is not a diffeomorphism
Differentiable Surface in 3-D

Definition: A subset $S \subset \mathbb{R}^3$ is a differentiable surface if every point $p$ in $S$ is contained in some open set $U$ of $S$ and there is a diffeomorphism $\phi : U \rightarrow W$ from $U$ to an open set of $\mathbb{R}^2$.

Note: Different $p$'s might require different $U$. 

$\phi$ is a standard calculus function.
Differentiable Surface in 3-D

\[ w_1 = \phi (x_1, x_2, x_3) \]
\[ w_2 = \phi (x_1, x_2, x_3) \]

\[ \phi^{-1} \]
\[ x_1 = \phi^{-1}(w_1, w_2) \]
\[ x_2 = \phi^{-1}(w_1, w_2) \]
\[ x_3 = \phi^{-1}(w_1, w_2) \]

\( \phi \) is a standard calculus function

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Co-ordinate function of the manifold
(Intrinsic co-ordinate)

Parametrization of the surface
(Local)
Differentiable Surface in 3-D

\[ \phi_1 \phi_2^{-1} \] and \[ \phi_2^{-1} \phi_1 \] are co-ordinate change functions

**Theorem 1:** S is a surface if and only if there exist diffeomorphic co-ordinate functions such that all co-ordinate change functions are diffeomorphisms

**Proof:** Depends on the differentiability of the inverse functions
The Intuitive idea of a manifold

Euclidean 3-d Space

Abstract Set

Surface

Manifold

If and only if

Use this as a definition

Theorem 1:
There exist homeomorphic co-ordinate functions such that
All co-ordinate change
functions are diffeomorphisms

What do we need to do this:

The notion of a co-ordinate function to a subset of

\( R^n \) for every point in the set.

(Manifold has to be a topological space)

Co-ordinate changes are diffeomorphisms

(These are only between subsets of \( R^n \))
Manifold
**Manifold: Definition 1**

**Definition:** A manifold $M$ is a topological Hausdorff space having a family of open sets $U_\alpha$ such that

1. Every point of $M$ belongs to some open set $U_\alpha$,
2. For every open set $U_\alpha$, there is a homeomorphism $\phi_\alpha : U_\alpha \rightarrow W_\alpha$ where $W_\alpha$ is an open subset of $\mathbb{R}$, and
3. Every co-ordinate change change function is a diffeomorphism,
4. Every open set having properties [1], [2], [3] is contained in the family $\{U_\alpha\}$.

Actually [1], [2], and [3] $\Rightarrow$ [4]

And you can construct a topology using the parametrization
Manifold

Note:

[1] All $W$ are open subsets in the $\mathbb{R}^n$, $n$ is fixed.
    $n$ is called the dimension of the manifold

[2] $S$ is no longer in a vector space, we have lost the capacity to add and substract element of $S$

   This has very serious implications for defining tangent vectors to $S$

[3] $S$ does not have a normal
The pair $U_\alpha$, $\phi_\alpha$ is called a *co-ordinate chart*.
The set of all co-ordinate charts is called an *atlas*.

Creating an atlas for a topological space is called

"putting a differential structure on the space."

**A deep question:** Are there incompatible differential structures

*on the same topological space?*

Yes!!
**Manifold: Definition 2**

*(do Carmo)*

**Definition:** A differentiable manifold of dimension $n$ is a set $M$ and a family of injective mappings $x_\alpha: U_\alpha \subset \mathbb{R}^n \rightarrow M$ if open sets $U_\alpha$ of $\mathbb{R}^n$ such that:

1. $\bigcup x_\alpha(U_\alpha) = M$,
2. For any pair $\alpha, \beta$ with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $x_\alpha x_\beta^{-1}$ are differentiable,
3. The family $\{(U_\alpha, x_\alpha)\}$ is maximal relative to the conditions [1] and [2].
Manifold: Definition 2

Notes: Define an subset $O$ of $M$ to be open if $x_\alpha (O \cap x_\alpha (U_\alpha))$ is open for all $\alpha$

This gives a topology for $M$ in which $x_\alpha$ are homeomorphisms to their image.

The functions $x_\alpha^{-1} x_\beta$ viewed as attachment functions define an identification space from the sets $U_\alpha$. This space is homeomorphic to the manifold
Manifold: Examples

Use all 6 projections to cover the sphere
Manifold: Examples

$P^2$: The set of all lines through the origin of the space

= The set formed by identifying diametrically opposite points on the surface of a 3-D sphere

The real projective space $P^n :=$ The set of all lines the origin of $\mathbb{R}^{n+1}$

Atlas for $P^2$

Use only 3 projections

2-D manifold
Each element of $\mathbb{P}^2$ is a line through the origin

The intersection of the line with the plane is a point in the plane

This defines a map from

$\mathbb{P}^2$ - lines through the equator $\rightarrow$ plane

The map is a homeomorphism

This defines a projective line: it is set of points in $\mathbb{R}^3$ which satisfy $Ax + By + Cz = 0$, for $C \neq 0$. 
Defn: A projective line is the set of points in $\mathbb{R}^3$ which satisfy $Ax + By + Cz = 0$, where $A, B, C$ are not all zero.

$\mathbb{P}^2 = \mathbb{R}^2 \cup \text{Projective Line (at infinity)}$

$\mathbb{P}^2$ is important in vision because it is exactly the action of a pin-hole camera.
Manifold: Examples

The ray manifold of $\mathbb{R}^3$: The set of all rays in 3-D

The ray manifold is 4-D

Proof?
A manifold is any mathematical set that can be continuously parameterized by subsets of $\mathbb{R}^n$. 