Further Topology in Plain English

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Keep this in mind:

Specifying open sets is equivalent to specifying convergent sequences
Key Point

Topological equivalences are very hard to grasp intuitively.
We need formal techniques for doing this:

[a] We need to identify useful ways in which
topological spaces appear (are created)
in applications

Product spaces, Identification Spaces, Covering Spaces etc.

[b] We need ways of calculating topological invariants
of such spaces
(It is often easier to determine when topological spaces are not equivalent, then when they are equivalent)

Homotopy and homology groups of spaces.
Subspace Topology

**Defn:** Let $B$ be a subset of a topological space $A$. The subspace topology on $B$ is the topology that $W$ is an open set in $B$ if and only if $W = B \cap Z$, where $Z$ is open in $A$.

**Warning:** $W$ may not be open in the topology of $A$
Subspace Topology

$B = \text{closed unit square in the plane}$

$W$ is not open in the plane but is open in $B$
**Product Spaces**

**Defn:** If $A$ and $B$ are sets, then their product $A \times B$ is the set

$$A \times B = \{ (a,b) \mid a \in A, \ b \in B \}$$

$$\Pi_1 : A \times B \rightarrow A \quad \Pi_1((a,b)) = a$$

$$\Pi_2 : A \times B \rightarrow B \quad \Pi_2((a,b)) = b$$

What we want:

A sequence $(a_n,b_n)$ is convergent in $A \times B$ if and only if $a_n$ is convergent in $A$ and $b_n$ is convergent in $B$.

A set $O \subseteq A \times B$ is open if and only if its projections on $A$ and $B$ are open.
**Product Spaces**

**Defn:** The product space $A \times B$ of two topological spaces is the set of all ordered pairs

$$A \times B = \{(a, b) \mid a \in A, b \in B\},$$

with the following system of open sets:

A subset $W$ of $A \times B$ is open if its projections on $A$ and $B$ are open.

**Theorem:** The projection functions are continuous
Product Spaces

How should we visualize this?

For a fixed \( a \in A \), the set \( \{(a,b), b \in B\} \) looks just like \( B \)

For a fixed \( b \in B \), the set \( \{(a,b), a \in A\} \) looks just like \( A \)

\[ R^2 = R \times R \text{ with the usual topologies on } R \]
Descarte’s brilliant idea

Take \( \mathbb{R}^2 \) with its usual topology

Take the plane with its usual topology

(Open sets are unions of open discs)

Impose a co-ordinate system on the plane

This gives a one-to-one and onto function \( f \) from \( \mathbb{R}^2 \) to the plane

\[ \begin{array}{c}
\ldots \\
(-1.0,-2.0) \\
\ldots \ldots \\
(0,0) \\
\ldots \ldots \\
(1.0,3.0) \\
\ldots \ldots \\
\end{array} \]

\[
\begin{tikzpicture}
  \draw[<->] (0,0) -- (0,4);
  \draw[<->] (0,0) -- (4,0);
  \draw[->] (0,0) .. controls (0.5,1.5) .. (1,2);
  \node at (1.5,1.5) {1-1 map};
\end{tikzpicture}
\]

\( f \) is not just one-to-one and onto but is actually a homeomorphism
**Identification Spaces**

Intuition: If you glue opposite sides of a square you get a cylinder.

Every point in the square goes to exactly one point in the cylinder.

Every point on the glued edge comes from a pair of points in the square.

Each element of this space is a subset of the square.

The new space is composed from disjoint subsets of the original space.
Identification Spaces

**Defn:** Let $X$ be a topological space

Let $P_{\alpha}$ be a family of disjoint subsets of $X$ such that

$$\bigcup_{\alpha} P_{\alpha} = X,$$

Let $Y$ be a set whose points are members of $P_{\alpha}$

Let $p: X \rightarrow Y$ be the map that takes every point of $X$

to the subset containing $Y$

Let a subset $O$ of $Y$ be open if and only if $p^{-1}(O)$ is

en open in $Y$

Under these conditions $Y$ is a topological space with the

*identification topology*. ($Y$ is an *identification space*)
Identification Spaces
Identification Spaces
Identification Spaces

An extension of this idea gives us a manifold
A really cool theorem

Let \( f: X \to Y \) be an onto and continuous function,
(\text{and suppose that } Y \text{ has the largest topology for which } f \text{ is continuous}), then \( f \) partitions \( X \) according to \( f^{-1}(y), \ y \in Y \).

The technical condition is satisfied if \( X \) is compact and \( Y \) is Hausdorff

Let \( Y* \) be the identification space associated with the partition.

**Theorem:** \( Y* \) is homeomorphic to \( Y \)
Level Sets

Level sets are connected exactly like $\mathbb{R}$!!
Level Sets

\[ f = \sin(2\pi x) \sin(2\pi y) \]
Level Sets

\[ f = \sin(2 \pi x) \sin(2 \pi y) \]

Level Set

Open Set

Another level set approaching the first
Level Sets

\[ f = \sin(2\pi x) \sin(2\pi y) \]

Another level set approaching the first
Level Sets

$f = \sin(2 \pi x) \sin(2 \pi y)$

Another level set approaching the first
Further Topology

Just as we generalized the notions of open sets and continuous functions, we can generalize the notions of connected and compact sets.

Connectivity is a topological invariant.

Invariants are important for showing when two topological spaces are not homeomorphic.

The key to doing all of this is to generalize common notions by using a set of formal properties.

- surface
- derivatives
- vector

Knowing which properties to use takes genius.