Differential Geometry of Curves

- local analysis: differential calculus.

- global analysis: influence of local properties on the behavior of the entire curve.

Parameterized Curve

Definition:
a (infinitely) differentiable *map* $\alpha : I \to R^3$ of an open interval $I = (a, b)$ of real line R into R^3.

- $\alpha(t) = (x(t), y(t), z(t))$.

- *tangent vector*: $\alpha'(t) = (x'(t), y'(t), z'(t))$.

- *trace*: the image set $\alpha(I) \subset R^3$.

Parameterized Curve

Remarks:

- the map α needs not to be one-to-one.
- α is *simple* if the map is one-to-one.
- distinct curves can have the same trace:

$$\begin{aligned} \alpha(t) &= (cos(t), sin(t)) \\ \beta(t) &= (cos(2t), sin(2t)) \end{aligned}$$

Regular Curve

Definition:

a parameterized curve $\alpha : I \to R^3$ is said to be regular if $\alpha'(t) \neq 0$ for all $t \in I$.

- for the study of curve, it is essential that the curve is regular.
- *singular point*: where $\alpha'(t) = 0$.

Arc Length

Definition:

given $t \in I$, the *arc length* of a regular curve $\alpha : I \to R^3$, from the point t_o, is

$$s(t) = \int_{t_o}^{t} |\alpha'(t)| dt$$

where

$$|\alpha'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

is the length of the vector $\alpha'(t)$.

- since $\alpha'(t) \neq 0$, $s(t)$ is a differentiable function of t, and $ds/dt = |\alpha'(t)|$.

- if the curve is arc length parameterized, then $ds/dt = 1 = |\alpha'(t)|$.

- conversely, if $|\alpha'(t)| \equiv 1$, then $s = t - t_o$.

Curves Parameterized by Arc Length

Definition:

let $\alpha : I \rightarrow R^3$ be a curve parameterized by arc length $s \in I$, the number $|\alpha''(s)| = k(s)$ is called the *curvature* of α at s.

- at point where $k(s) \neq 0$, the *normal vector* $n(s)$ in the direction of $\alpha''(s)$ is well defined by $\alpha''(s) = k(s)n(s)$.

- the plane determined by $\alpha'(s)$ and $n(s)$ is called the *osculating plane*.

- *binormal vector*: $b(s) = t(s) \times n(s)$

Curves Parameterized by Arc Length

Definition:

let $\alpha : I \to R^3$ be a curve parameterized by arc length s such that $\alpha''(s) \neq 0, s \in I$, the number $\tau(s)$ defined by $b'(s) = \tau(s)n(s)$ is called the *torsion* of α at s.

- since $b'(s) = t'(s) \times n(s) + t(s) \times n'(s) = t(s) \times n'(s)$
  hence, $b'(s)$ is normal to $t(s)$, and is parallel to $n(s)$, and we may write
  $b'(s) = \tau(s)n(s)$

- if α is a plane curve, then the plane of the curve agrees with the osculating plane, hence $\tau = 0$.

- conversely, if $\tau \equiv 0$ and $k \neq 0$,
  $b(s) = b_o = constant$, and therefore
  $(\alpha(s) \bullet b_o)' = \alpha'(s) \bullet b_o = 0$
  it follows that $\alpha(s) \bullet b_o = constant$, and hence $\alpha(s)$ is contained in a plane normal to b_o.

Frenet Trihedron

To each value of the parameter s, there are three orthogonal unit vectors $t(s), n(s), b(s)$. The derivatives, called *Frenet Formulas*, are

$$\begin{aligned} t'(s) &= kn \\ b'(s) &= \tau n \\ (n'(s) &= b'(s) \times t(s) + b(s) \times t'(s) = -\tau b - kt) \end{aligned}$$

when expressed in the basis $\{t, n, b\}$, yield geometrical entities (curvature and torsion) about the behavior of α in a neighborhood of s.

- *rectifying plane*: tb plane.

- *normal plane*: nb plane.

- *principal normal*: line which contain $n(s)$ and pass through $\alpha(s)$.

- *binormal*: line which contain $b(s)$ and pass through $\alpha(s)$.

Fundamental Theorem

Given differentiable functions $k(s) > 0$ and $\tau(s), s \in I$, there exists a regular parameterized curve $\alpha : I \to R^3$ such that s is the arc length, $k(s)$ is the curvature, and $\tau(s)$ is the torsion of α. Moreover, any other curve $\bar{\alpha}$ satisfying the same conditions differs from α by a rigid motion.

- for plane curve, it is possible to give the curvature k a sign: under the basis $\{t(s), n(s)\}$, k is defined by

$$dt/ds = kn$$

- given a regular parameterized curve $\alpha : I \to R^3$ (not necessary parameterized by arc length), it is possible to obtain a curve $\beta : J \to R^3$ parameterized by arc length which has the same trace as α. (this all the extension of all local concepts to regular curves with an arbitrary parameter).

Local Canonical Form

Natural Local Coordinate system: the Frenet trihedron.

Taylor expansion:

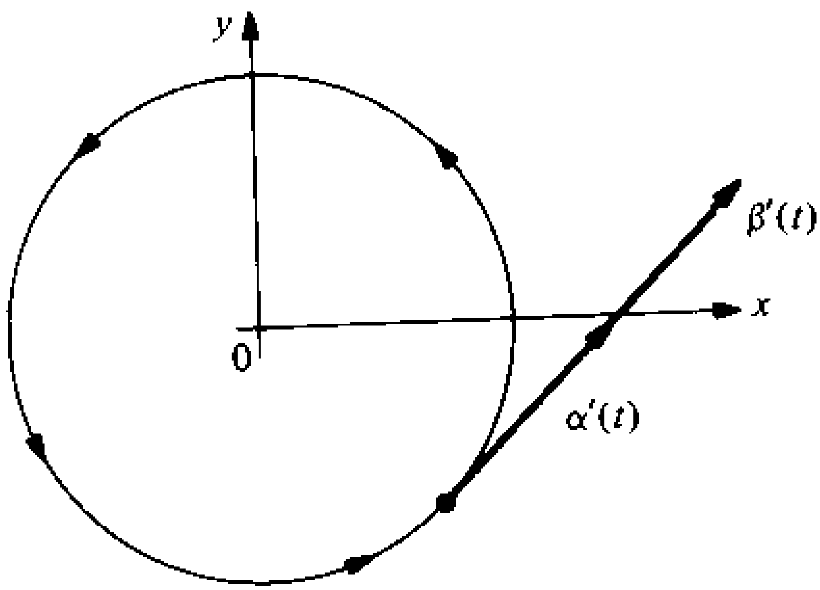
$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{s^2}{2}\alpha''(0) + \frac{s^3}{6}\alpha'''(0) + R(1)$$

since $\alpha'(0) = t, \alpha''(0) = kn, \alpha'''(0) = (kn)' = k'n - k^2t - k\tau b$, we have

$$\alpha(s) - \alpha(0) = (s - \frac{k^2s^3}{6})t + (\frac{s^2k}{2} + \frac{s^3k'}{6})n - k\tau b + R$$

where all terms are computed at $s = 0$. For $\alpha(t) = (x(t), y(t), z(t))$,

$$\begin{aligned} x(s) &= s - \frac{k^2s^3}{6} + R_x \\ y(s) &= \frac{k}{2}s^2 + \frac{k's^3}{6} + R_y \\ z(s) &= -\frac{k\tau}{6}s^3 + R_z \end{aligned}$$


y
β′(t)
x
0
α′(t)

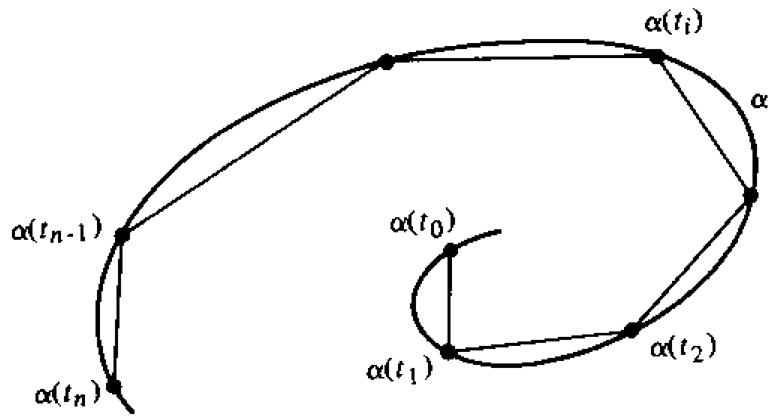
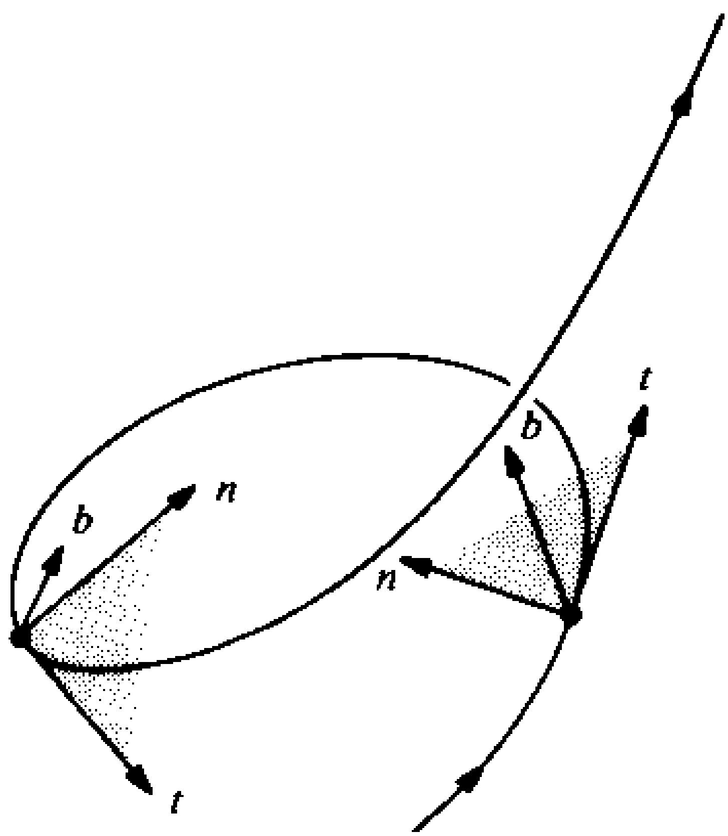
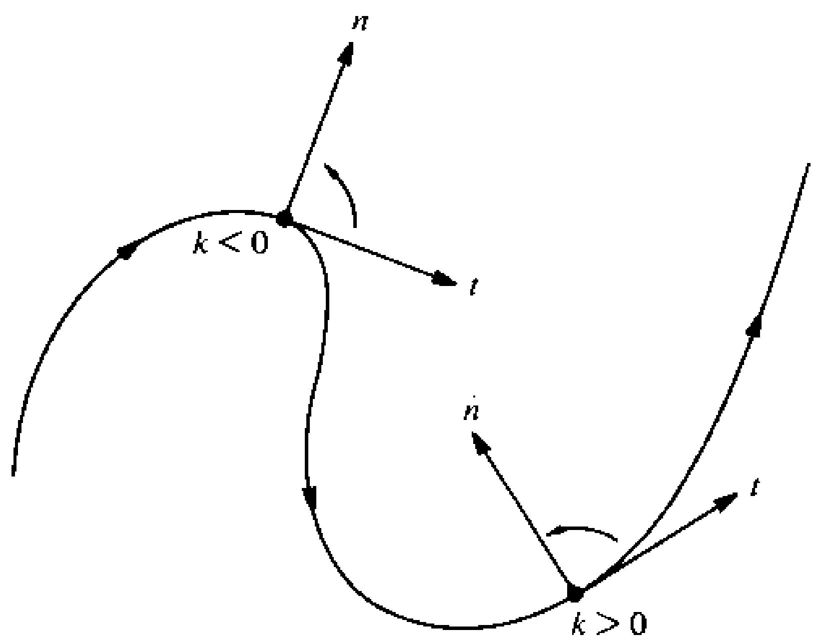


Figure 1-12



t
b
b
n
n
t



n
k < 0
t
n
t
k > 0