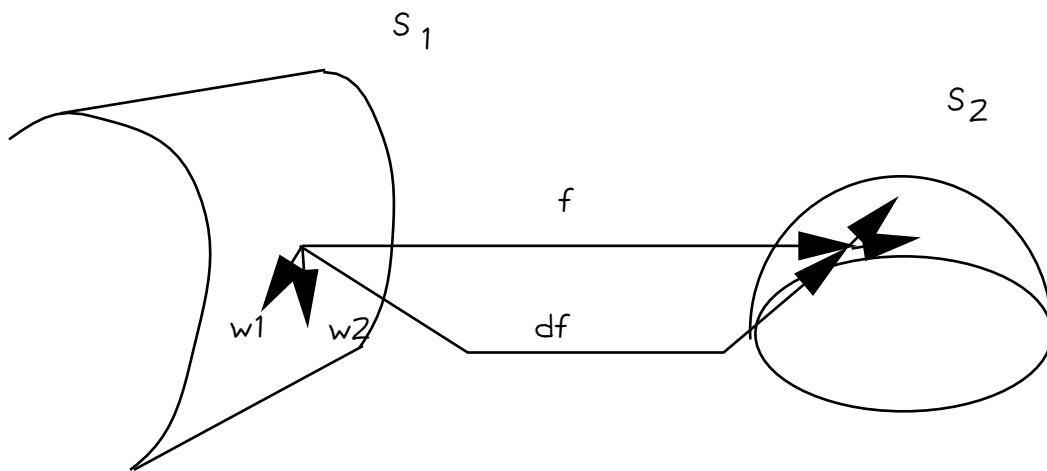


EE913a: Advanced Topics in Medical Imaging  
and Computer Vision

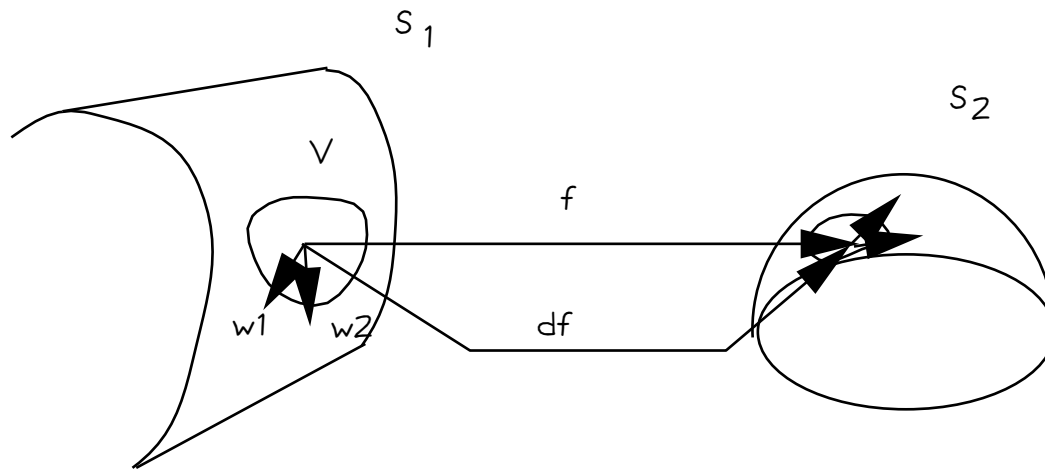
Note Set No. 13

# Isometries



Defn: A diffeomorphism  $f: S_1 \rightarrow S_2$  is an isometry if for all  $p$  in  $S$   
and all pairs of vectors  $w_1, w_2$  in  $T_p(S_1)$   
 $\langle w_1, w_2 \rangle_p = \langle df(w_1), df(w_2) \rangle_{f(p)}$

# Local Isometries

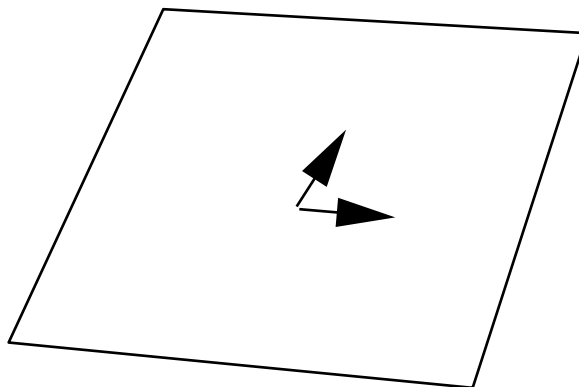
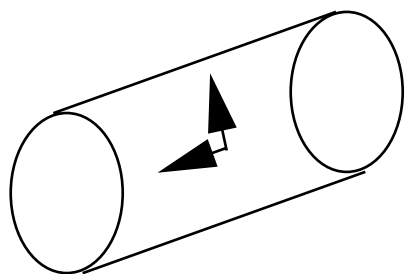


Defn: A map  $f: V_1 \rightarrow S_2$  of a neighborhood  $V_1$  of  $p$  in  $S_1$  is a local isometry at  $p$  if there exists a neighborhood  $V_2$  of  $f(p)$  in  $S_2$  such that  $f: V_1 \rightarrow V_2$  is an isometry. If there exists a local isometry from  $S_1$  to  $S_2$  at every point of  $S_1$  then  $S_1$  is said to be locally isometric to  $S_2$ .

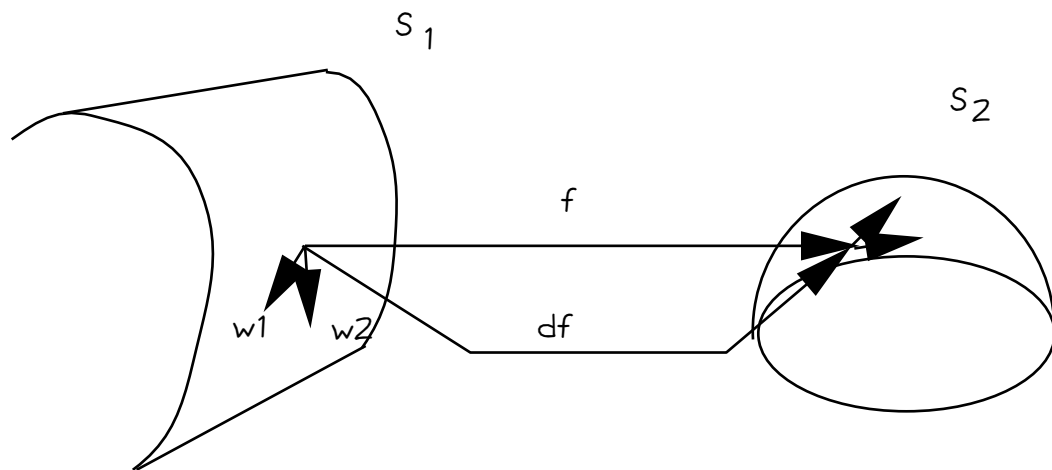
# Local Isometries

Two surfaces can be locally isometric without being globally isometric.

Example. Plane and a cylinder.



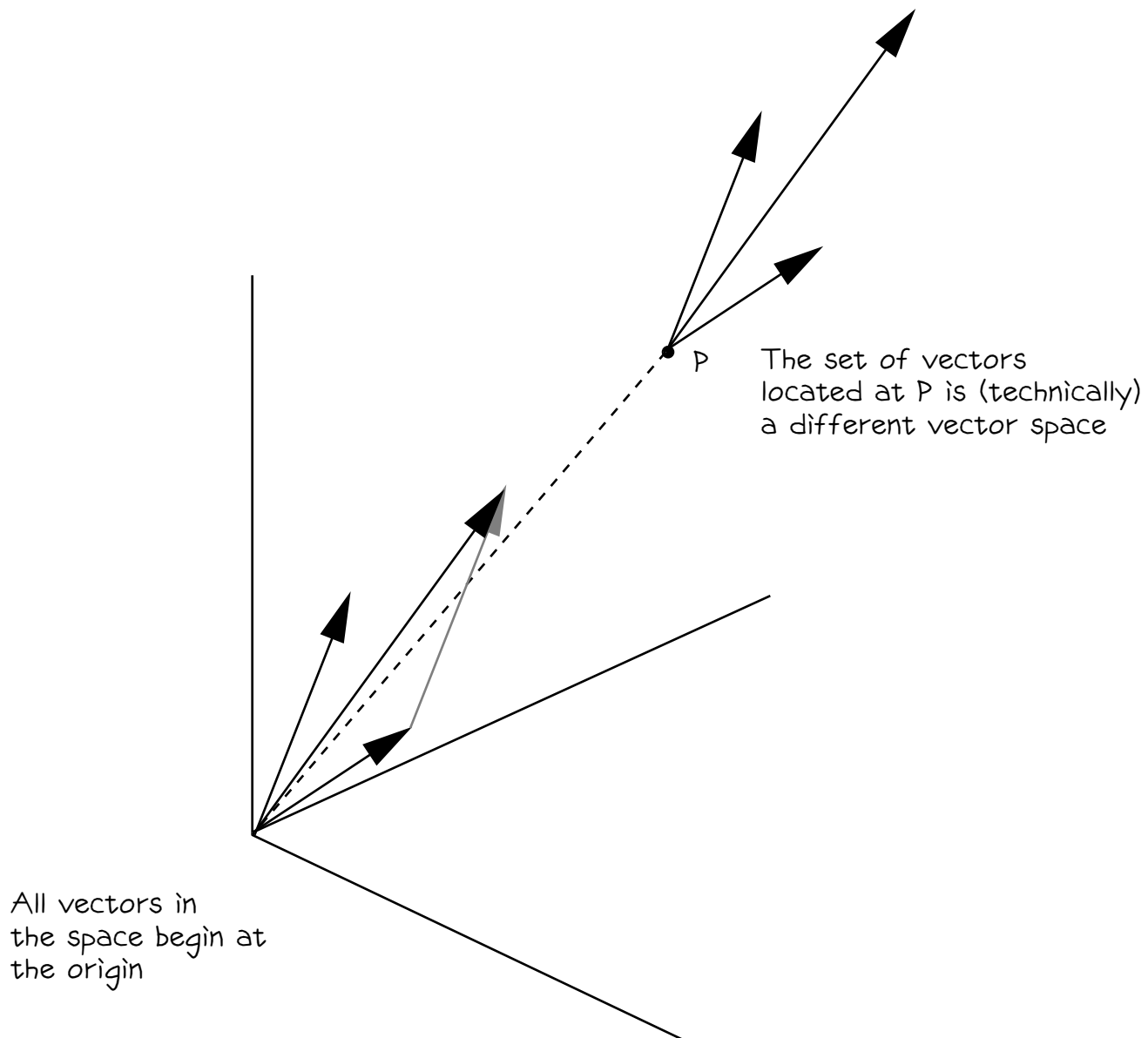
# Gauss' Theorema Egregium



Theorem: The Gaussian curvature of a surface is invariant to local isometries.

Proof: Let  $x:U \subset \mathbb{R}^2 \rightarrow S_1$  be a parameterization of  $S_1$  at  $p$ . Since  $f:V_1 \subset S_1 \rightarrow S_2$  is a diffeomorphism,  $f \circ x: U \rightarrow S_2$  is a local parameterization of  $S_2$  at  $f(p)$ . Since  $f$  is an isometry, the coefficients of the first fundamental form in  $V_1$  and  $f(V_1)$  agree. Hence the Gaussian curvature is invariant.

## More Intrinsic Geometry: Parallel Vectors Along a Line



How does this generalize to surfaces?

# The Axiomatic Approach: One way to generalize a mathematical idea

Simple structure

Complex structure

?

Class of objects  
(sets or functions)  
e.g. vectors, parallel transport  
Laplacian

Definitions to not  
generalize in an obvious  
way (definitions may be  
coordinate dependent)

These may be quite strange  
and very non-intuitive at first

Properties that uniquely  
define the object

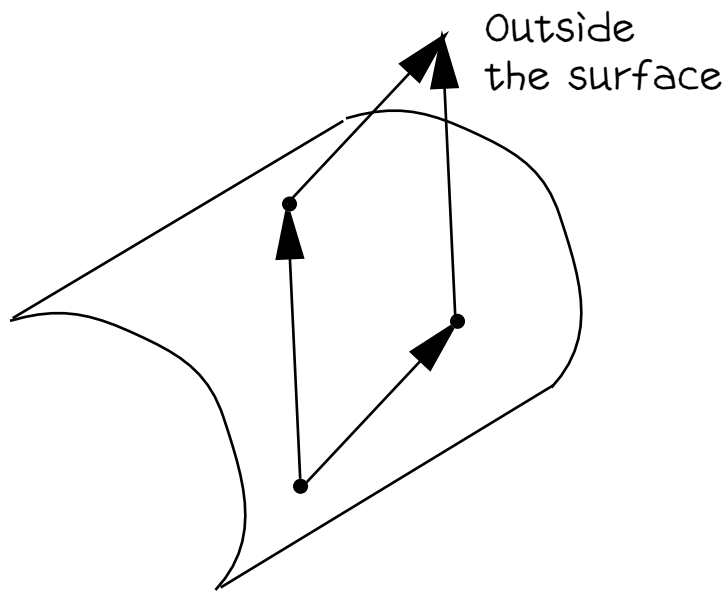
Axioms

Any object that possesses  
these properties (satisfies  
the axioms) is a generalization  
of the concept.

## Examples:

Vector spaces (generalize algebraic properties of real numbers)  
Metric Spaces (generalize properties of distance)  
Topology (generalizes the notion of limits)

## Vectors on the surface

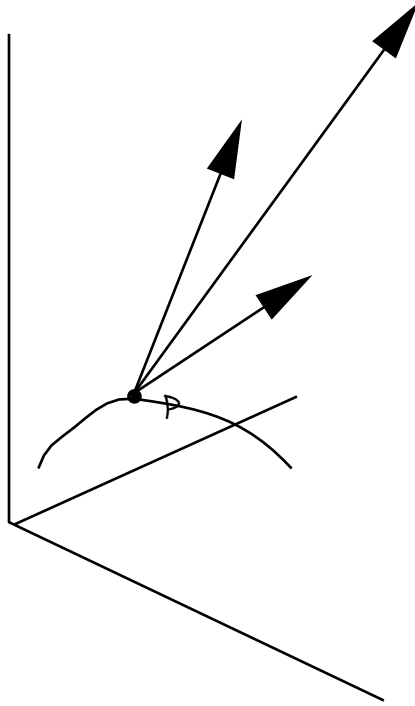


We want to define vectors intrinsically -- that is only using points on the surface without a notion of the surrounding space.

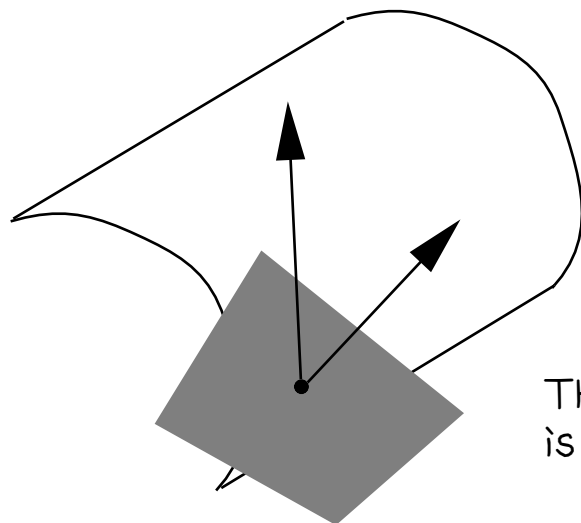
However we define the vectors we want all vectors based at a point to form a vector space (closed under addition and multiplication by real numbers).

Defining a vector as an "arrow" from one point to another on a surface does not work (adding two such vectors does give another arrow from one point of the surface to another).

# Vectors on the surface

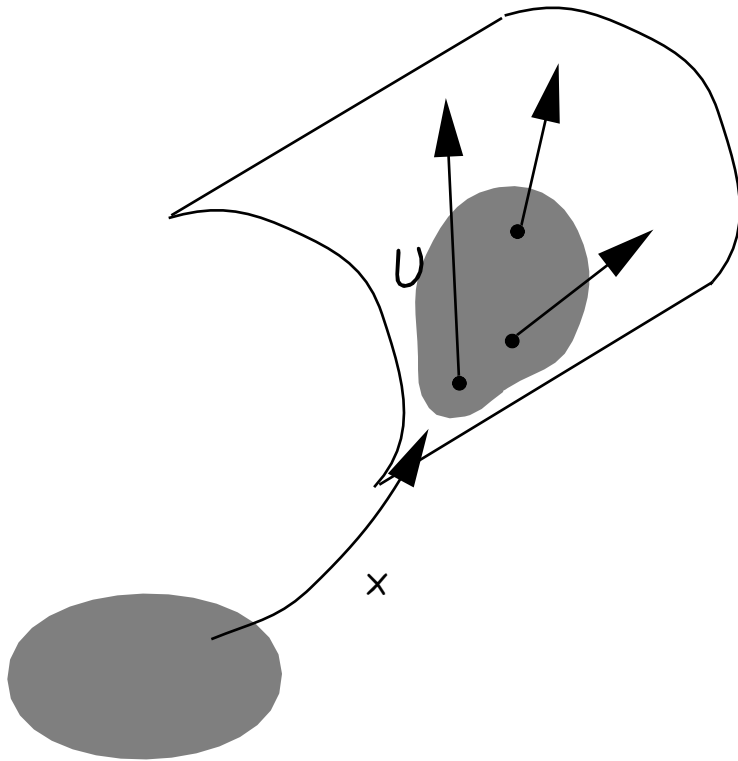


The set of vectors based at a point  $p$  is identical to the set of tangent vectors to curves passing through that point



The set of all vectors is the tangent space  $T_p$

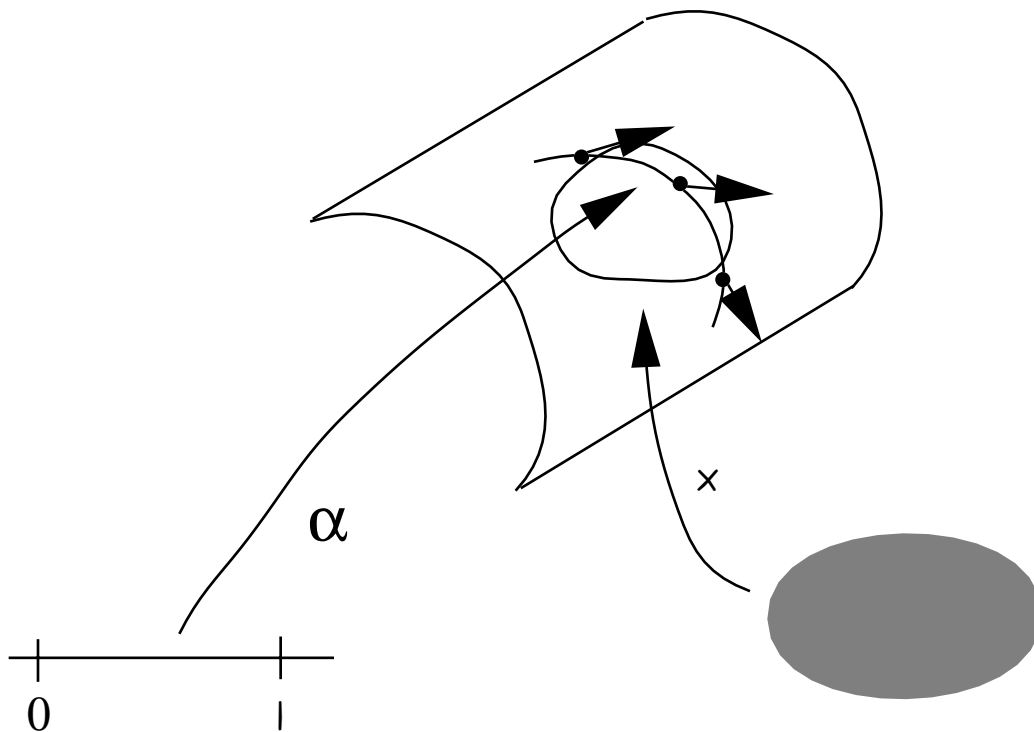
## Vector field on the surface



Definition: A vector field on an open set  $U$  of a surface  $S$  is a function (correspondence)  $w$  that assigns to each point  $p$  of  $U$  a vector  $w(p) \in T_p$ .

Definition: A vector field  $w$  is differentiable at  $p$  if, for some parameterization  $x(u,v)$  in  $p$ , the components  $a$  and  $b$  of  $w = ax_u + bx_v$  are differentiable functions of  $p$ .

## Vector field along a curve on the surface



Definition: A parametrized curve  $\alpha: [0,1] \rightarrow S$  is the restriction to  $[0,1]$  of a differentiable mapping of  $(0-\varepsilon, 1+\varepsilon)$ ,  $\varepsilon > 0$ , into  $S$ . If  $\alpha(0)=p$  and  $\alpha(1)=q$ , we say that  $\alpha$  joins  $p$  to  $q$ . The curve is regular if  $\alpha'(t) \neq 0$  for  $t$  in  $[0,1]$ .

Definition: Let  $\alpha: I \rightarrow S$  be a parameterized curve in  $S$ . A vector field  $w$  along  $\alpha$  is a function (correspondence) that assigns to each  $t$  in  $I$  a vector  $w(t)$  in  $T_w(t)$ . The vector field is differentiable at  $t_0$  in  $I$  if for some parameterization  $x(u,v)$  around  $\alpha(t_0)$  the components  $a(t)$  and  $b(t)$  of  $w(t) = a(t)x_u + b(t)x_v$  are differentiable functions of  $t$  at  $t_0$ . The vector field  $w$  is differentiable in  $I$  if it is differentiable for every  $t$  in  $I$ .

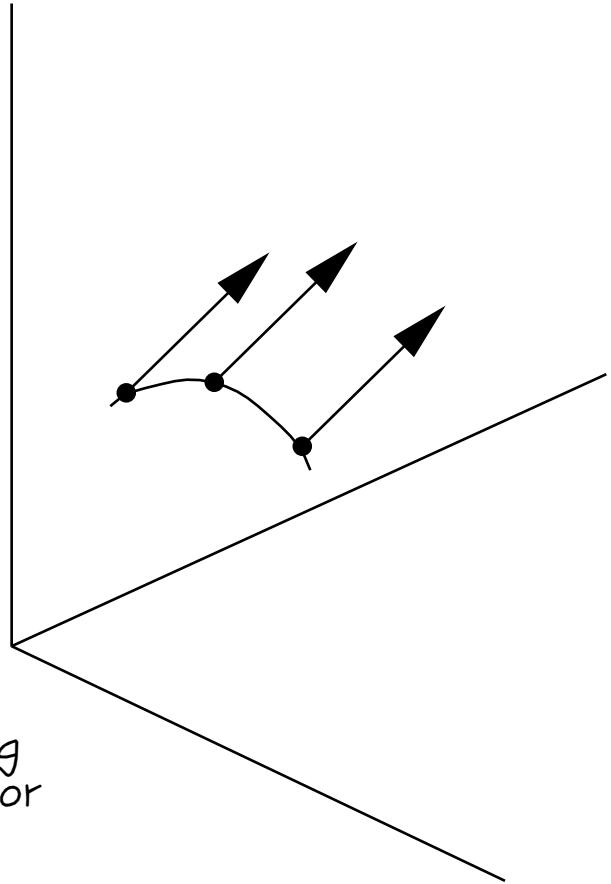
## Key Idea

A vector field along a curve in  $\mathbb{R}^3$  is parallel if the derivative of the field along the curve is zero.

The vector is parallel transported along a curve if its derivative is zero.

Note: The derivative of a vector along a curve at  $p$  is a vector in the vector space of vectors located at  $p$ .

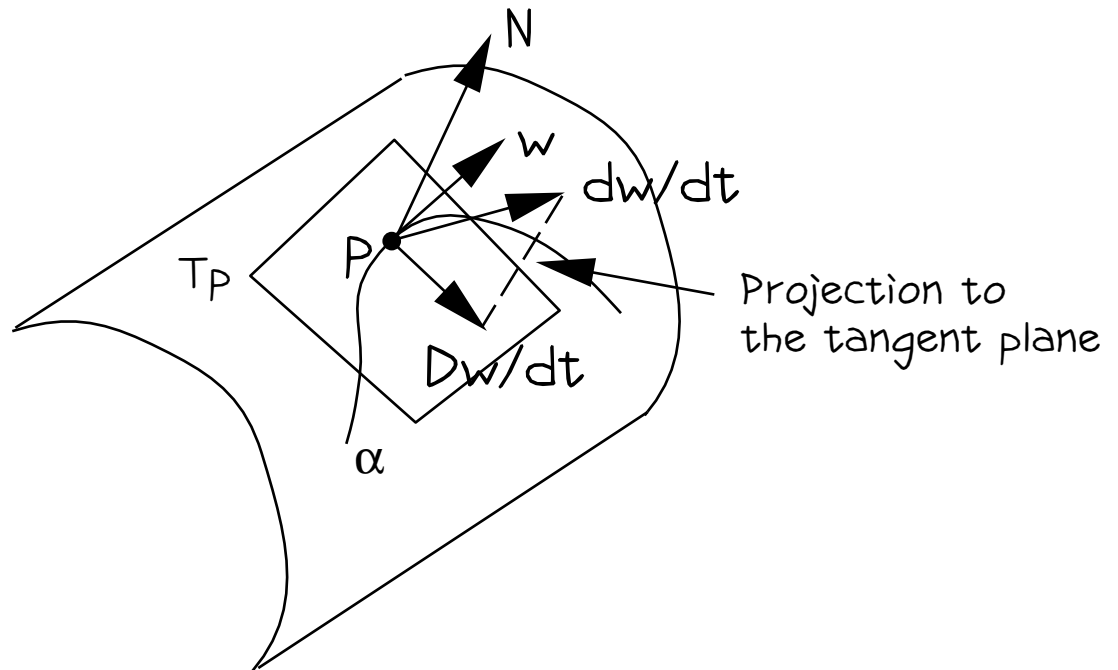
Slight abuse of the word "parallel"



We need to generalize the notion of the derivative of a vector field along a curve to surfaces. This generalization must give the derivative at  $p$  of a vector field on a surface as a vector in the tangent space at  $p$ .

This generalization is called a covariant derivative.

# Covariant Derivative

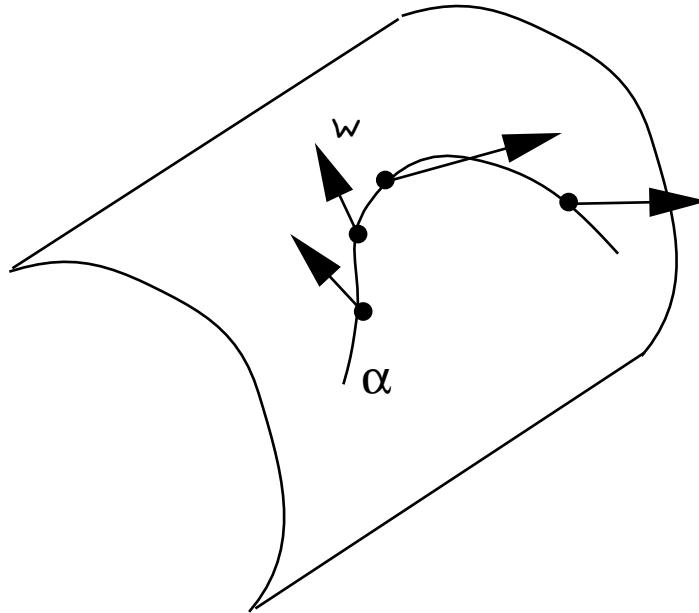


Definition: Let  $w$  be a differentiable vector field in an open set  $U$  of surface  $S$  and let  $p$  be a point in  $U$ . Let  $\alpha$  be a regular curve on  $S$  such that  $\alpha(0)=p$  and  $\alpha'(0)=y$ . Let  $w(t)$  be the restriction of  $w$  to  $\alpha$ . Then the projection of  $dw/dt(0)$  onto the tangent plane  $T_p(S)$  is the covariant derivative at  $p$  of  $w$  relative to the vector  $y$  (in the direction of  $y$ ). The covariant derivative is denoted  $Dw/dt(0)$ , or as  $(D_y w)(p)$ .

Notes:

- (1) The covariant derivative gives a vector in the tangent space  
Hence as an operator it is "intrinsic."
- (2) The covariant derivative is well defined for a vector field  
along a curve. We will use it as such.

## Parallel Vector Field Along a Curve

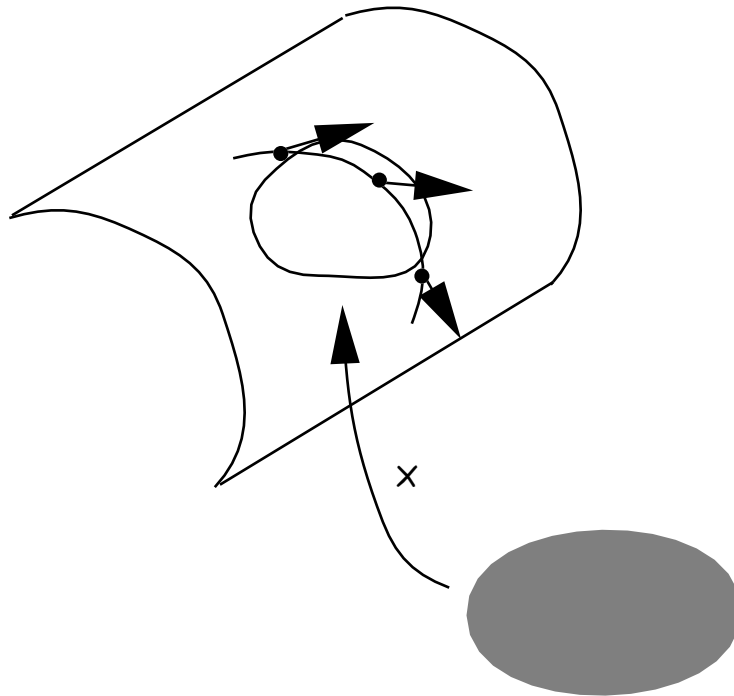


Definition: A vector field along a curve is parallel if its covariant derivative is zero at all points of the curve

$$Dw/dt = 0 \text{ for all } t.$$

This generalizes the notion of a parallel vector field.

# Calculations



Let the curve be  $\alpha(t) = (u(t), v(t))$ . Then the vector field  $w$  restricted to the curve is

$$\begin{aligned} w(t) &= a(u(t), v(t))x_u + b(u(t), v(t))x_v \\ &= a(t)x_u + b(t)x_v \end{aligned}$$

$$\begin{aligned} dw/dt &= a(x_{uu} u' + x_{uv} v') + a'x_u + b(x_{uv} u' + x_{vv} v') + b'x_v \\ &= (a' + \Gamma_{11}^1 a u' + \Gamma_{12}^1 a v' + \Gamma_{12}^1 b u' + \Gamma_{22}^1 b v') x_u \\ &\quad + (a' + \Gamma_{11}^2 a u' + \Gamma_{12}^2 a v' + \Gamma_{12}^2 b u' + \Gamma_{22}^2 b v') x_v + \text{term} * N \end{aligned}$$

$$\begin{aligned} Dw/dt &= a(x_{uu} u' + x_{uv} v') + a'x_u + b(x_{uv} u' + x_{vv} v') + b'x_v \\ &= (a' + \Gamma_{11}^1 a u' + \Gamma_{12}^1 a v' + \Gamma_{12}^1 b u' + \Gamma_{22}^1 b v') x_u \\ &\quad + (a' + \Gamma_{11}^2 a u' + \Gamma_{12}^2 a v' + \Gamma_{12}^2 b u' + \Gamma_{22}^2 b v') x_v \end{aligned}$$