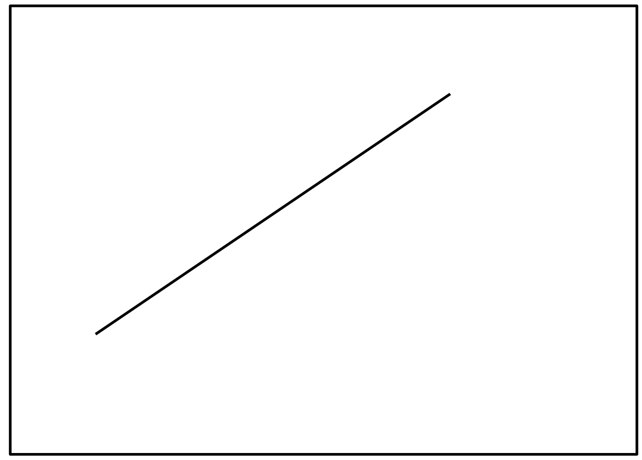
EE913a: Advanced Topics in Medical Imaging and Computer Vision

Note Set No. 16

Geodesics

A straight line is an important special curve in the plane.

We want to generalize the notion of straight line to surfaces.

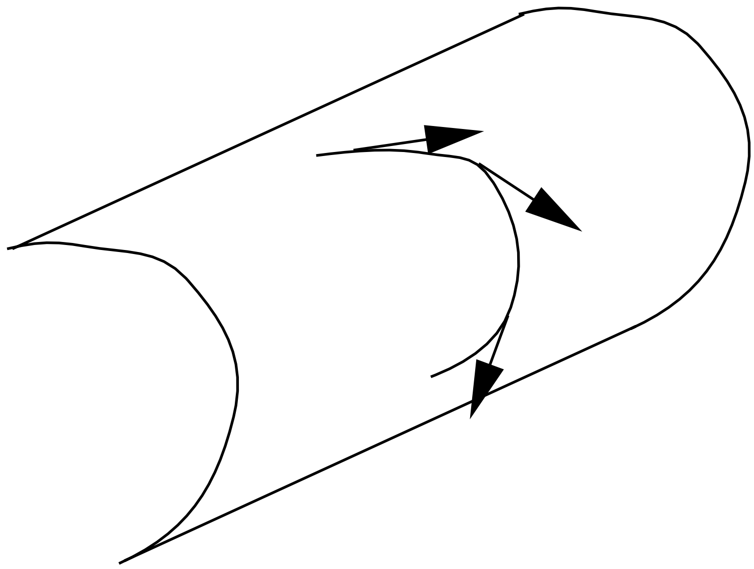


Differential characterization of a strainght line in a plane.

A straight line in the plane is a curve whose tangent vectors form a parallel field.

A "straight line" a.k.a. "geodesic" on a surface is a curve whose tangent vectors form a parallel field.

Geodesic



Definition: A non-constant parametrized curve $\gamma: I \to S$ is said to be a geodesic at t in I if the field of its tangent vectors $\gamma'(t)$ is parallel along γ at t, i.e.

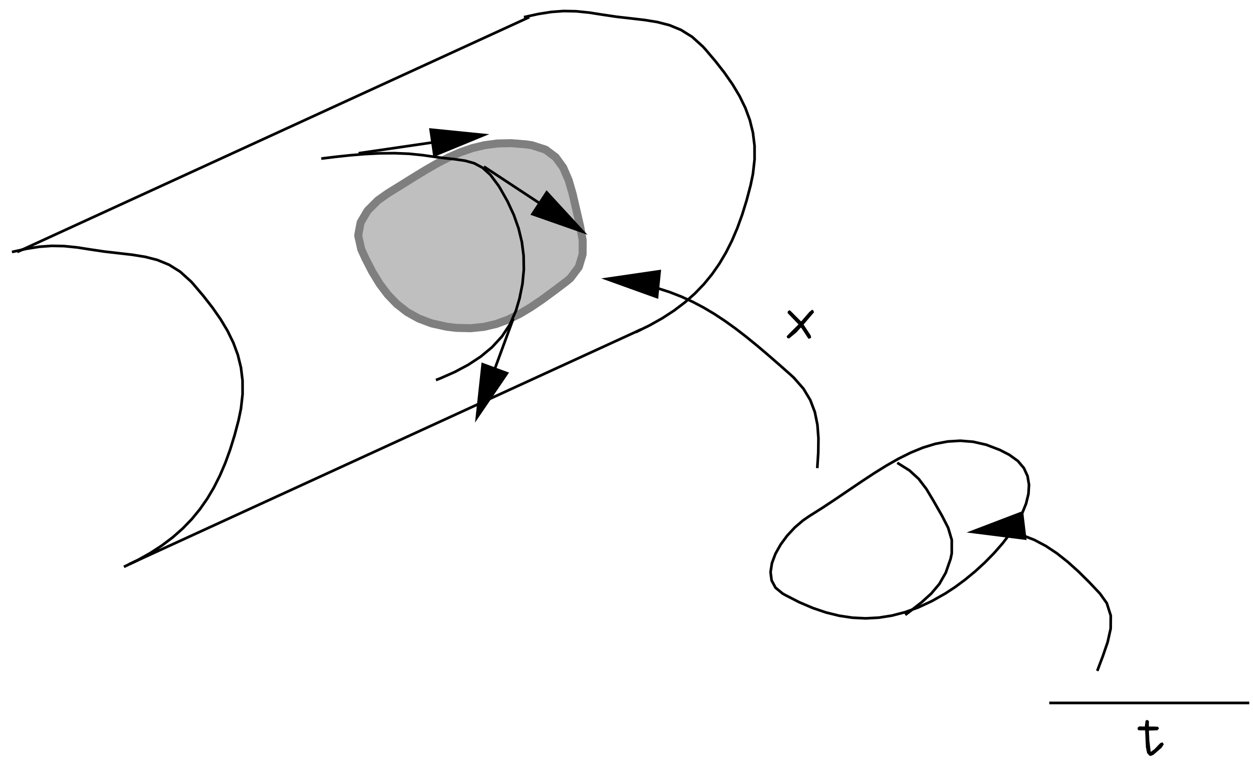
$$D\gamma'(t)/dt = 0,$$

γ is a parametrized geodesic if it is a geodesic for all t in I.

Note: [1] For a geodesic γ, we have $|\gamma'(t)|$ = constant $= c \neq 0$. Thus we may introduce the arc length = ct as a parameter.

[2] The notion of a geodesic is local property that is invariant under isometries of surfaces.

Equations for a Geodesic



Let x(u,v) be a parametrization of surface S in the neighborhood of some point p in S. Let J be an open interval and X(u(t),v(t)) be a curve γ on S. Then the tangent vector field on the curve is

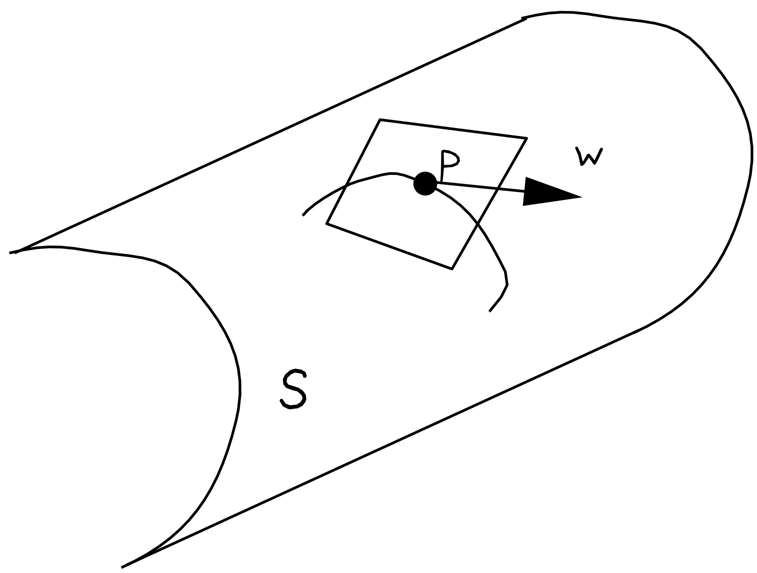
$$w = u'(t)x_u + v'(t)x_v.$$

Setting Dw/dt=0, gives,

$$u'' + \Gamma^1 11(u')^2 + 2\Gamma^1 12u'v' + \Gamma^1 22(v')^2 = 0,$$
$$v'' + \Gamma^1 11(v')^2 + 2\Gamma^1 12u'v' + \Gamma^1 22(v')^2 = 0.$$

Second order non-linear coupled differential equations for the "parameters" u,v of the geodesic.

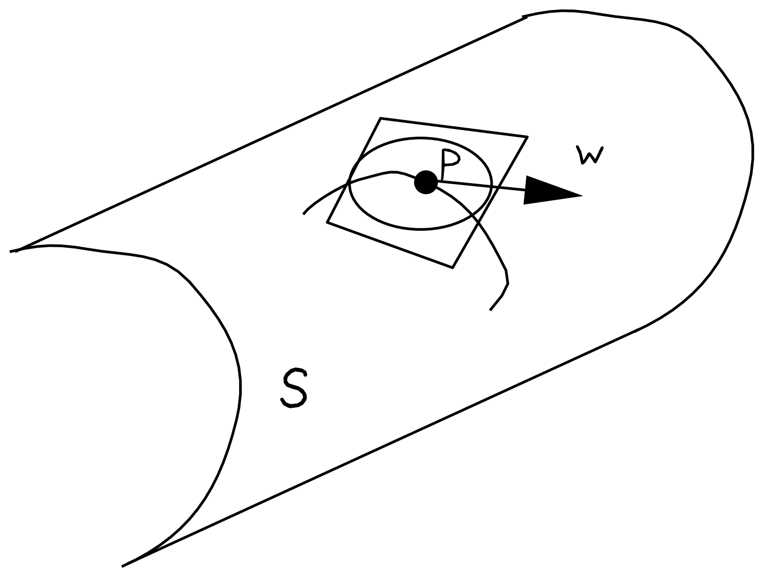
Existence and Uniqueness of a Geodesic



Proposition: Given a point p in S and a vector $w \neq 0$ in $T_p(S)$, there exists an $\varepsilon > 0$ and a unique parametrized geodesic $\gamma:(-\varepsilon,\varepsilon) \to S$ such that $\gamma(0)=p$, and $\gamma'(0)=w$.

Note: No bounds on ε are given by this theorem.

Uniform Existence



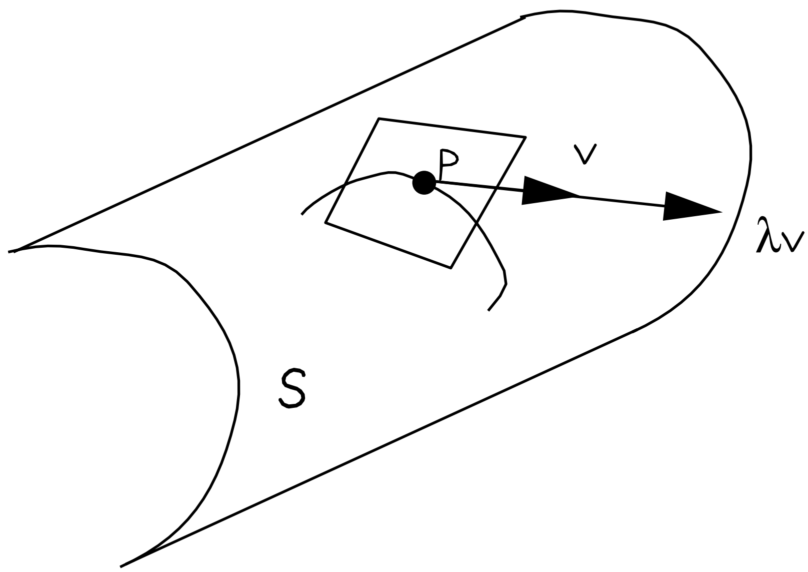
Theorem: Given p in S, there exist numbers $\varepsilon 1 > 0$ and $\varepsilon 2 > 0$ and a differentiable map

$$\gamma: (-\varepsilon 2, \varepsilon 2) \times B_{\varepsilon 1} \to S$$

such that for all v in $B_{\varepsilon 1}$, v != 0, t in $(-\varepsilon 2, \varepsilon 2)$, the curve $\gamma(t,v)$ is a geodesic with $\gamma(0)=p$, and $\gamma'(0)=v$, and for $v=0$, $\gamma(t,0)=p$.

Note: No bounds on $\varepsilon 2$ are given by this theorem.

Geodesics of Scaled Vectors



<u>Lemma</u>: If the geodesic $\gamma(t,v)$ is defined for t in $(-e,e)$ then the geodesic $\gamma(t,\lambda v)$, λ != 0, is defined for t in $(-e/\lambda, e/\lambda)$, and $\gamma(t,\lambda v) = \gamma(\lambda t, v)$.

<u>Proof</u>: Define $a:(-e/\lambda, e/\lambda)$ -> S by $a(t) = \gamma(\lambda t, v)$. Then $a(0)=p$ and $a'(0)=\lambda v$. Further, by linearity of the covariant derivative $D_{a'(t)}\, a'(t) = \lambda^2\ D_{\gamma'(t,v)}\, \gamma'(t,v) = 0$.

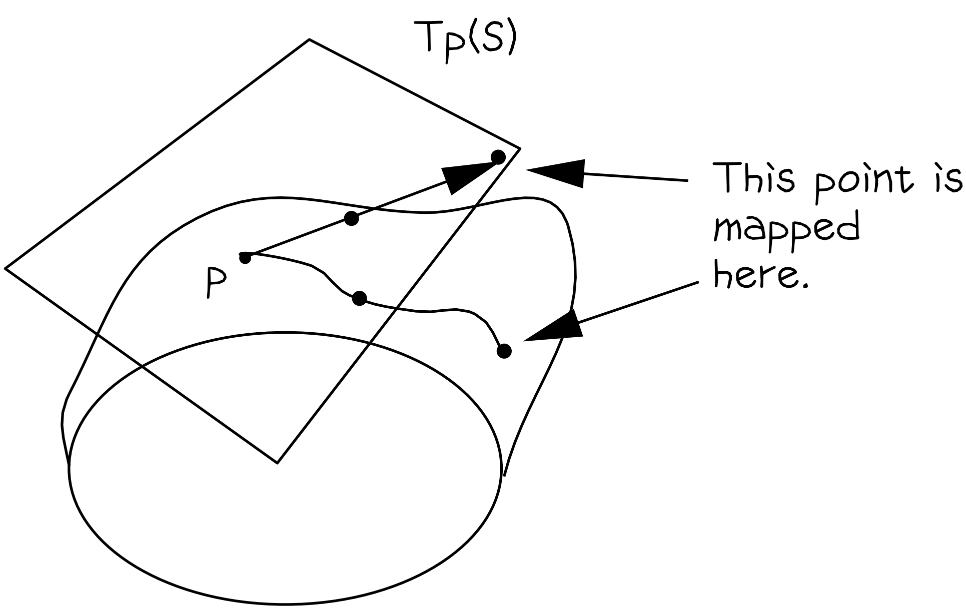
The Exponential Map

<u>Definition</u>: If v in $T_p(S)$, v!=0 is such that $\gamma(|v|,v/|v|) = \gamma(1,v)$

is defined, set

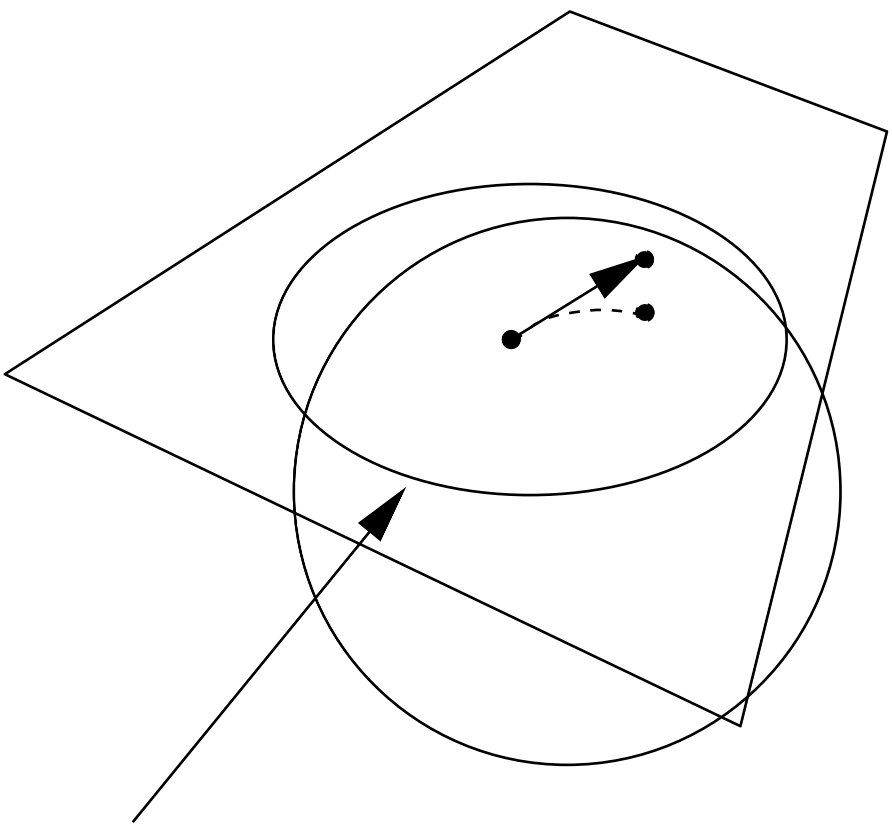
$\exp_p(v) = \gamma(1,v)$, and $\exp_p(0) = p$.

This is called the exponential map.



The Exponential Map

Example: Sphere



Sphere radius=1.

Circles of radii π, 3π, ... $(2n+1)\pi$ are mapped to single point.

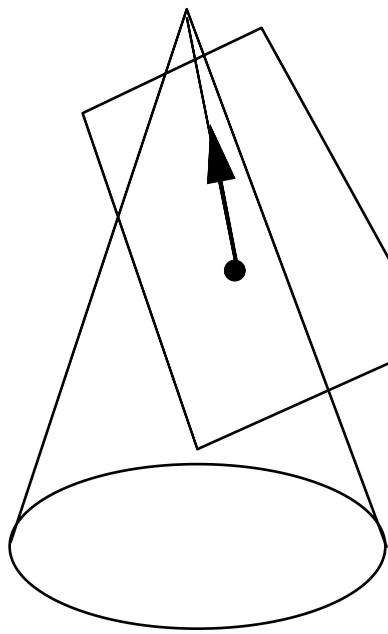
Note: The exponential map need not be 1-to-1

Question: Can you think for a surface for which it is 1-to-1

and onto?

The Exponential Map

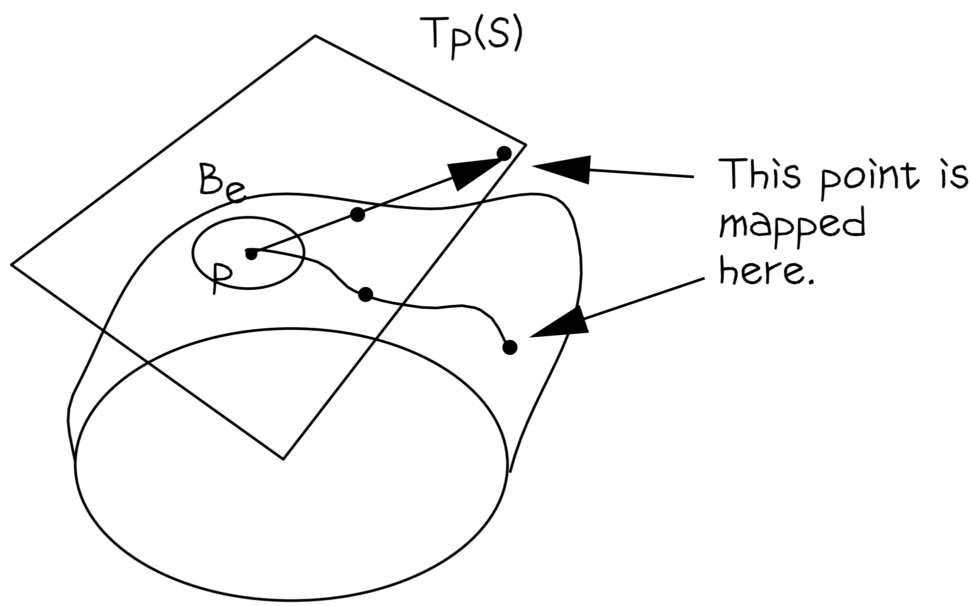
Example: Cone - apex



The exponential map is not defined in the line joint a point to the apex for distances >= the distance of the point to the apex.

Note: The exponential map may not be defined for the entire tangent plane.

Properties of the Exponential Map



Proposition 1: Given p in S, there exists an e > 0 such that exp_p is defined and differentiable in the interior B_e of a disk of radius e of $T_p(S)$ with center in the origin.

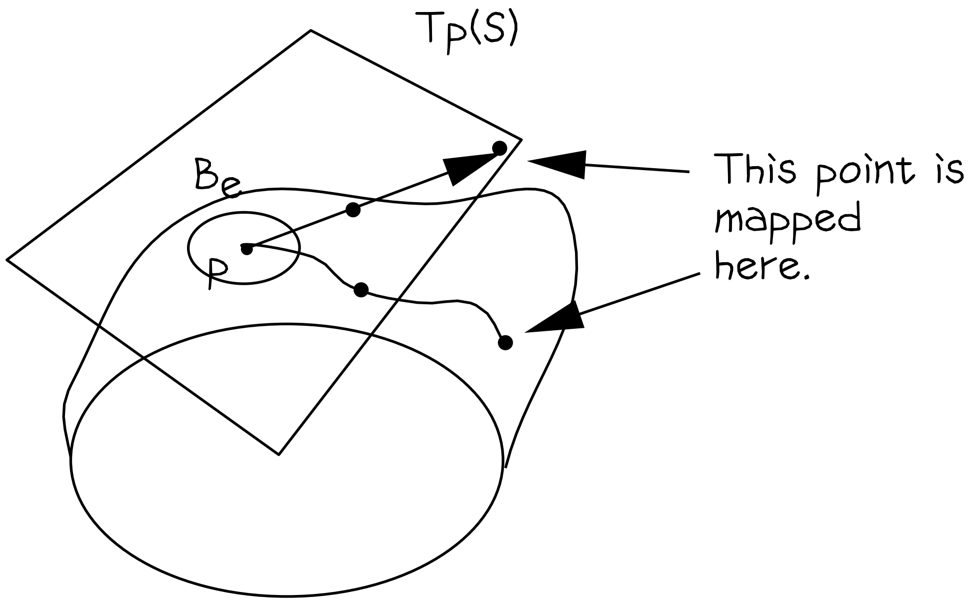
Recall:

Theorem: Given p in S, there exist numbers $\varepsilon 1 > 0$ and $\varepsilon 2 > 0$ and a differentiable map

$$\gamma: (-\varepsilon 2, \varepsilon 2) \times B_{\varepsilon 1} \rightarrow S$$

such that for all v in $B_{\varepsilon 1}$, v != 0, t in $(-\varepsilon 2, \varepsilon 2)$, the curve $\gamma(t,v)$ is a geodesic with $\gamma(0)=p$, and $\gamma'(0)=v$, and for v=0, $\gamma(t,0)=p$.

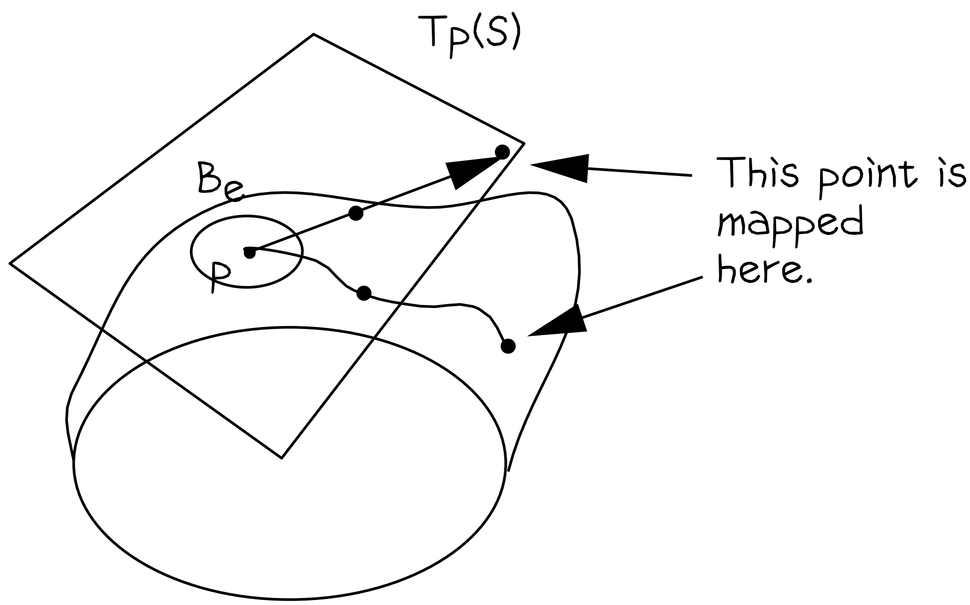
Properties of the Exponential Map



Proposition 1: Given p in S, there exists an $e > 0$ such that exp_P is defined and differentiable in the interior B_e of a disk of radius e of $T_P(S)$ with center in the origin.

Proof: Since $\gamma(t,v)$ is defined for $|t| < e2$ and $|v| < e1$, we set $\lambda=e2/2$, and know that $\gamma(t,\lambda v)$ is well defined for $|t|<2$ and $|v| < e1$. Thus, setting the radius of B_e to be $< e1.e2/2$ we have that $\gamma(1,w)= exp_P\ (w)$ is well defined for w in B_e.

Properties of the Exponential Map



Theorem: The exponential map is a diffeomorphism in a neighborhood U of B_e.

Proof: We will show that $d(exp_p)$ is not singular at the origin, the result follows from the inverse function theorem.

Think of T_p as a surface and exp_p as a map from T_p to S. The tangent plane to T_p at 0 is T_p.

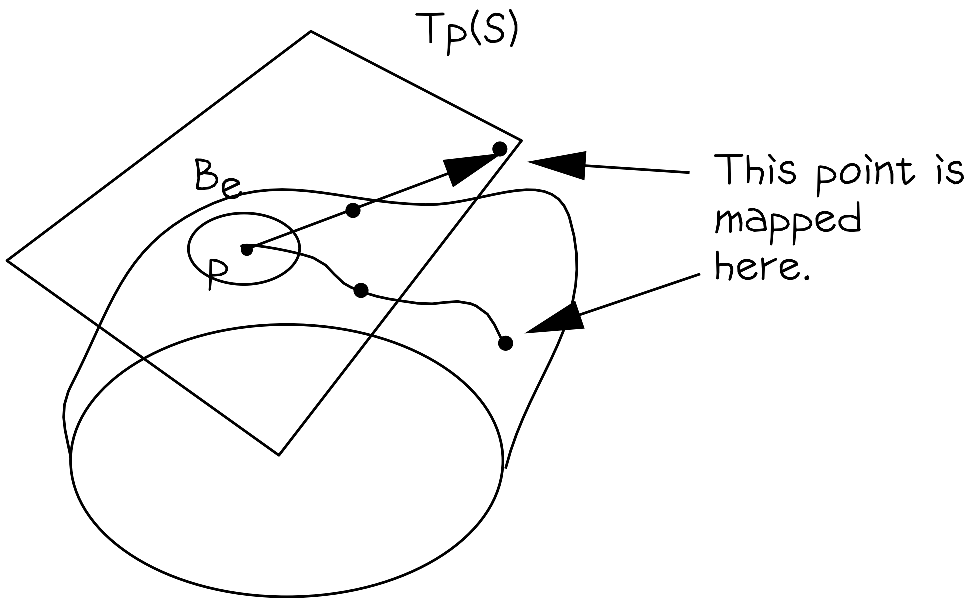
Let a(t) = tv, v!=0, be a curve in T_p. Its image onto S is $exp_p(a(t)) = exp_p(tv)$.

Then $d/dt\ exp_p(a(t))$ at t=0

$= d/dt\ exp_p(tv) = d/dt\ \gamma(1,tv) = d/dt\ \gamma(t,v)$ at t=0

= v.

Properties of the Exponential Map



Summary:

[1] The map is not neccesarily 1-to-1

[2] It may not be defined for all vectors in T_p

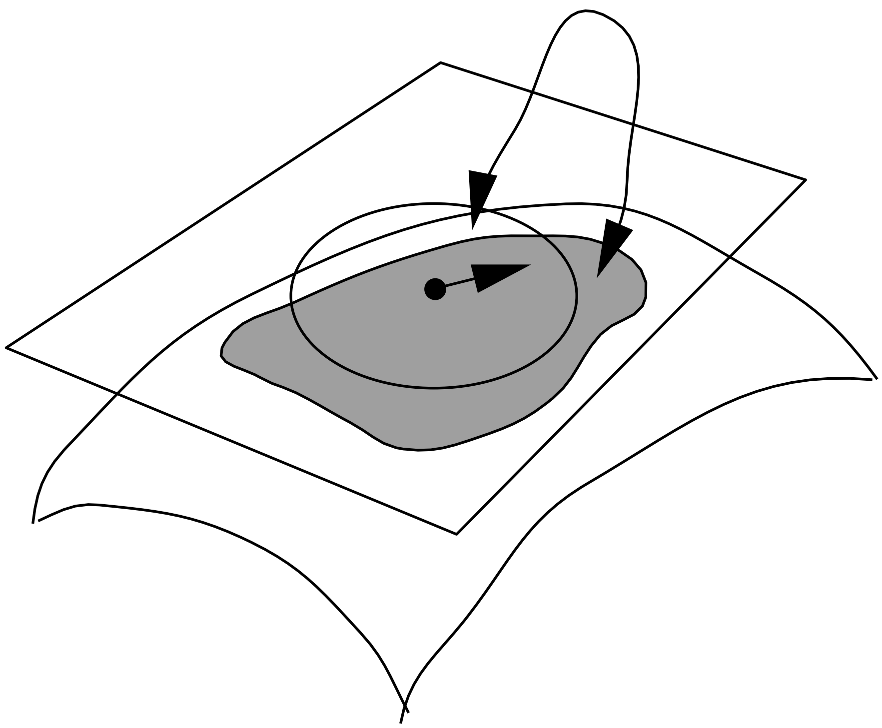
[3] But, it is well defined in the interior of a disk at the origin

[4] And in a neigbborhood of 0, it is a diffeomorphism with the differential map being the identity operator.

This neighborhood is called a Normal Neighborhood of p.

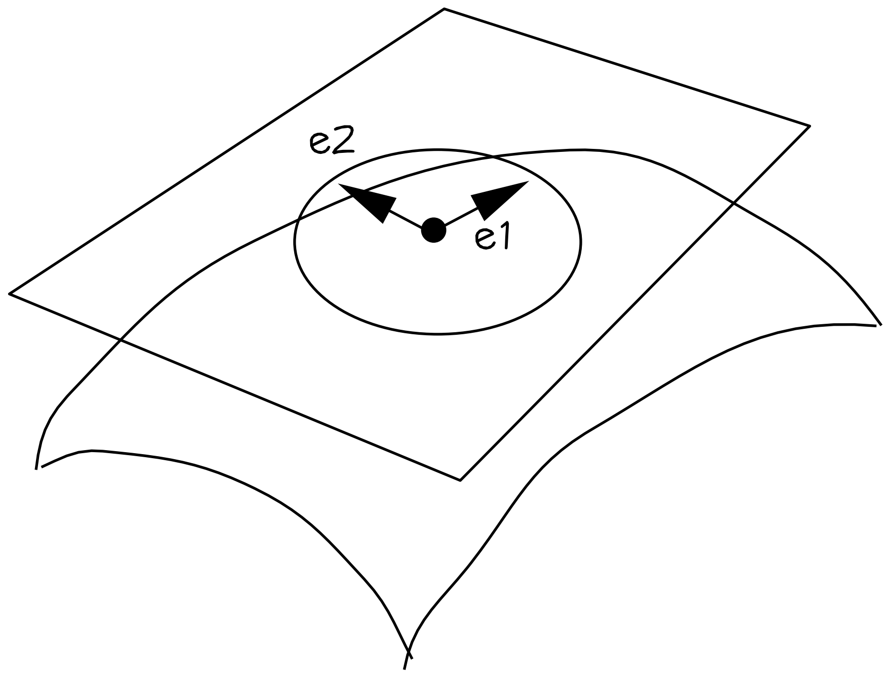
Normal Neighborhood

The exponential map maps the normal neighborhood diffeomorphically onto the surface



We can use the normal neighborhood to introduce canonical coordinate systems on a surface

Normal Coordinates



[1] Choose two orthonormal vectors in $T_p(S)$

any vector w in $T_p(S)$ can be written as
$w = u1e1 + u2e2$.

[2] Since $exp_p: U \rightarrow V$ in S is a diffeomorphism $exp_p(u1e1+u2e2)$

is a parametrization of V.

In this parametrization, the coefficients of the first fundamental form at p are $E(p) = G(p) = 1$, $F(p) = 0$.

Geodesic Polar Coordinates

(Informal Introduction)