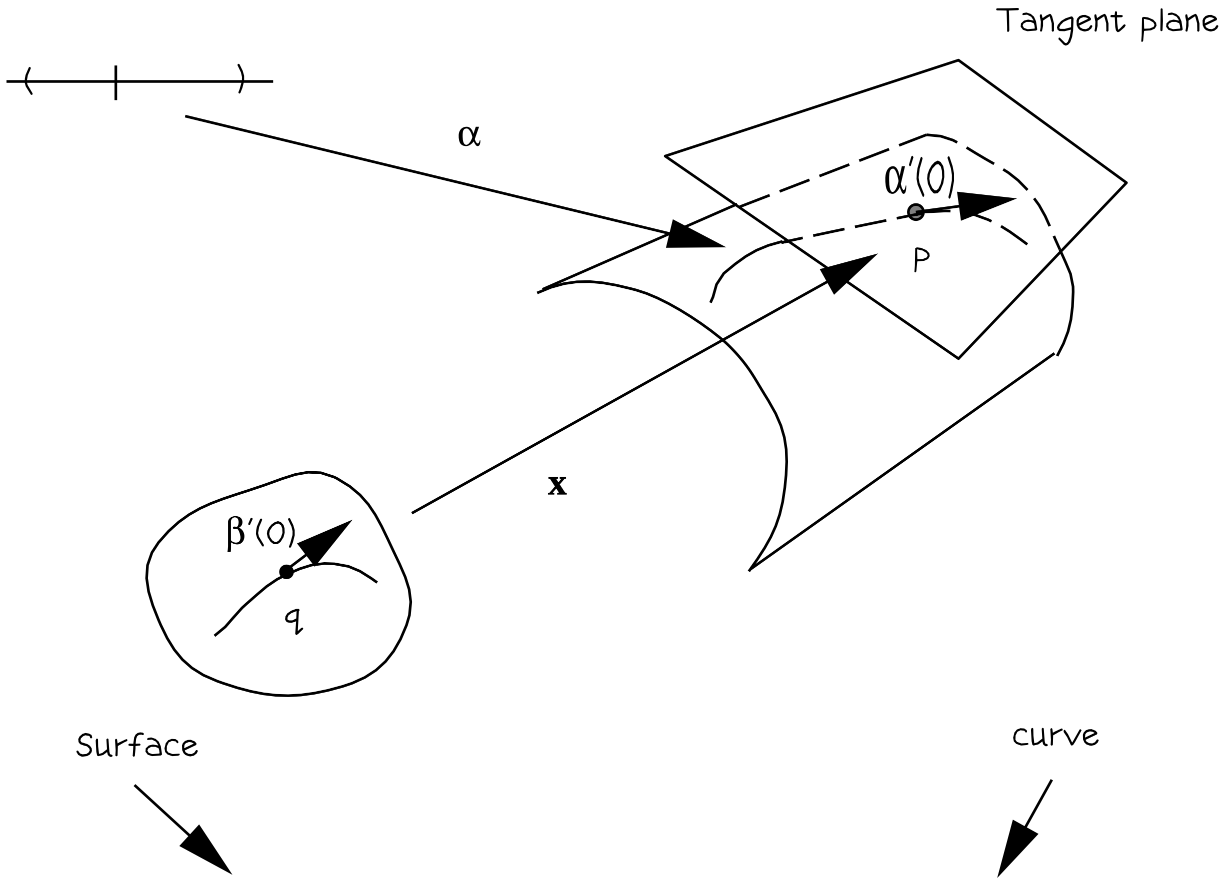
EE913a: Advanced Topics in Medical Imaging and Computer Vision

Note Set No. 6

The Tangent Plane

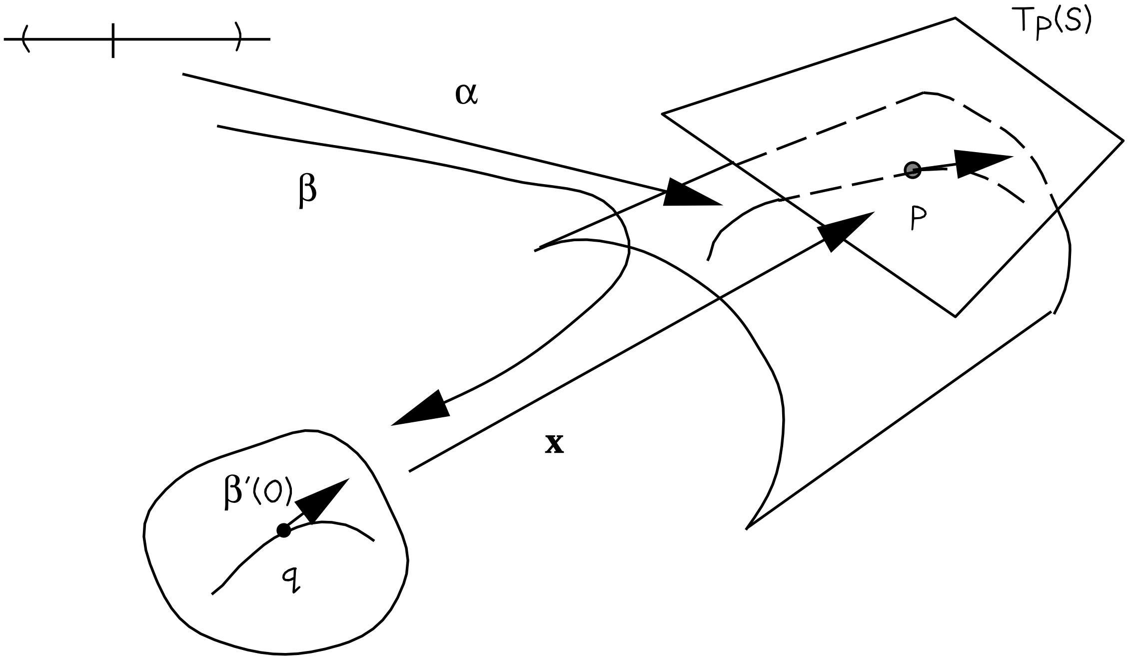


Defn: The tangent vector to a surface S at p in S is a tangent vector of (some) a curve α: (−e,e) -> S.

The set of all tangent vectors at p is called the tangent space to S at p.

Theorem 5.1: Let $\mathbf{x}$: U in R^2 ->S be a parmeterization of surface with $\mathbf{x}(q) = p$. Then the vector (sub-) space of dimension 2, $d\mathbf{x}q(R^2)$ in R^3 is the tangent space of S at p.

The Tangent Plane

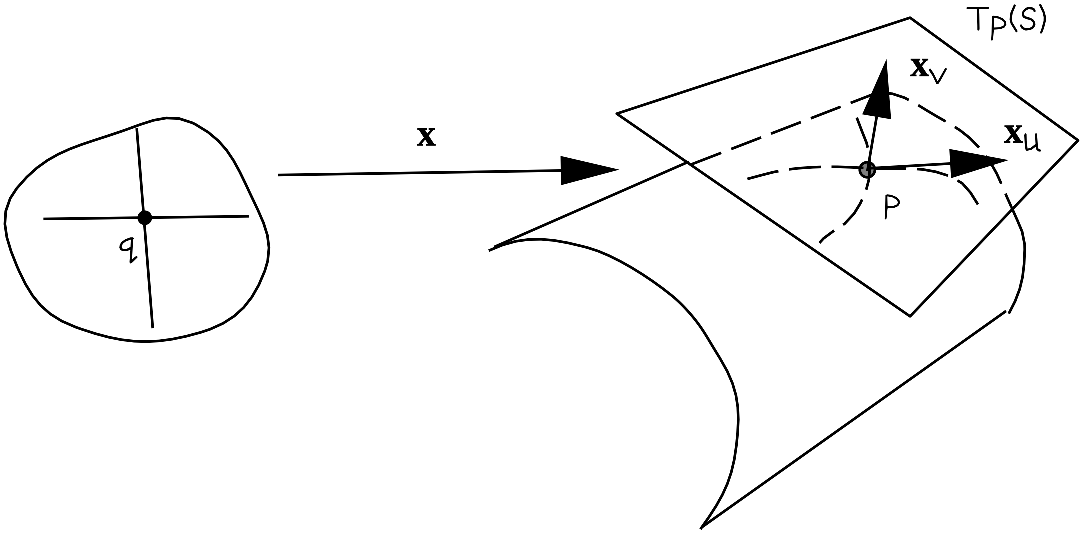


Note: The result is independent of the parameterization $\mathbf{x}$.

Proof: Consider the curve $\beta = \mathbf{x}^{-1} \circ \alpha$. Note, $\mathbf{x} \circ \beta = \alpha$.
So that $d\mathbf{x}_q(\beta'(0)) = \alpha'(0)$. Thus, every tangent vector is contained in $d\mathbf{x}_q(R^2)$. Conversely for every vector w at q, we can construct a curve which has w as its tangent vector. By the definition of a differential, $d\mathbf{x}_q(w)$ is a tangent vector at p. QED.

Notation: The tangent plane at p is denoted $T_p(S)$.

The Tangent Plane



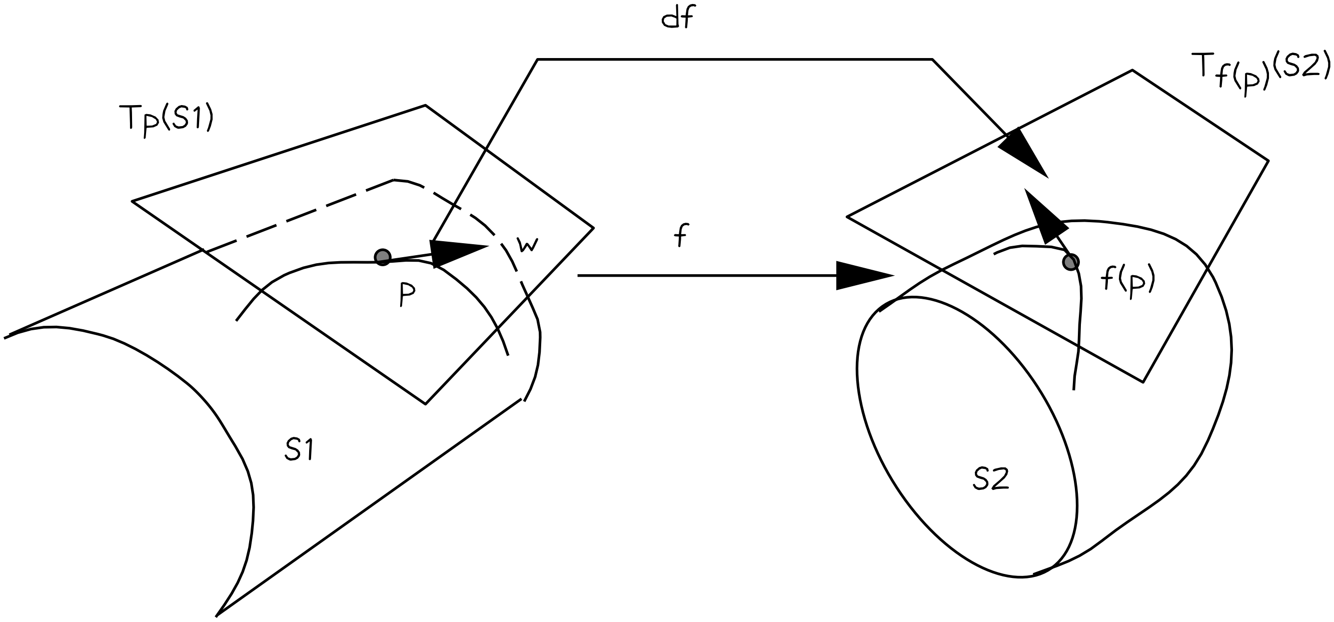
Note: Given a parameterization the vectors

$$d\mathbf{x}/du = (\ dx/du\ \ dy/du\ \ dz/du\ ) = \mathbf{x}_u$$

and $$d\mathbf{x}/dv = (\ dx/dv\ dy/dv\ dz/dv\ ) = \mathbf{x}_v$$ form a basis for $T_P(S)$.

Thus the representation of $T_P(S)$ with the basis vectors that depend on the coordinate function that we choose. But the definition of $T_P(S)$ is independent of this.

Differential of a map between surfaces



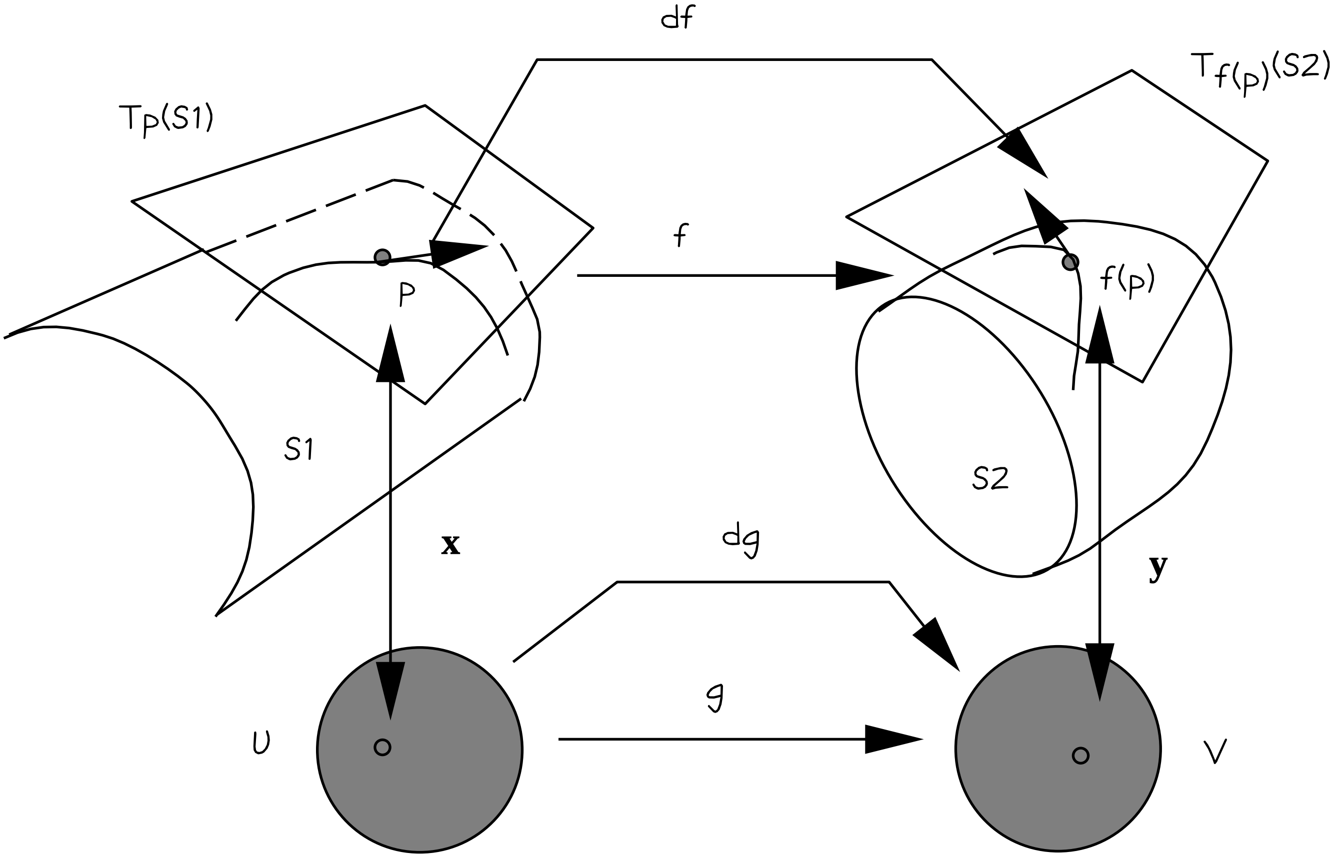
Defn: Let S1 and S2 be two regular surfaces and let $f: V$ in S1 -> S2 be a differentiable map of an open set V of S1. For a point p in V, and w in $T_p(S1)$ we know that there is curve α such that $\alpha(0)=p$ and $\alpha'(0)=w$. Then, $\beta = f o \alpha$ is a curve in S2 with a tangent vector in in $T_{f(p)}(S2)$. The map df_p: $T_p(S1) \to T_{f(p)}(S2)$ given by

$$df_p(w) = \beta'(0)$$

is called the differential of the map f.

Theorem: The differential df_p is a linear map and its definition is independent of the choice of the curve α.

Differential of a map between surfaces



g: U -> V $\quad g = \mathbf{y}^{-1} \circ f \circ \mathbf{x}$

g is differentiable ('cause x,y,f are differentiable)

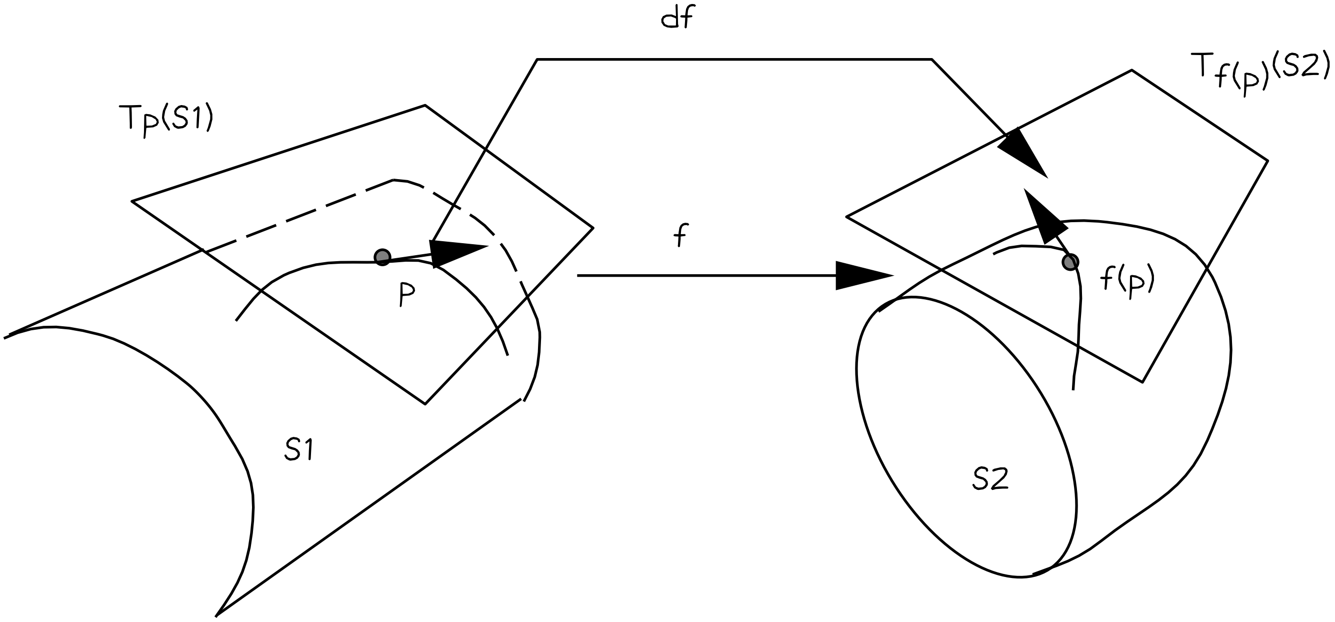
the representation of dg in the standard coordinate system is

also the representation of df.

Let g(u,v) = (g1(u,v),g2(u,v))

then the matrix representation of dg is $\begin{bmatrix} dg1/du & dg1/dv \\ dg2/du & dg2/dv \end{bmatrix}$

Inverse function theorem between surfaces

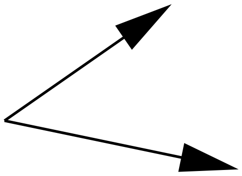


<u>Theorem 5.2:</u> If S1 and S2 are regular surfaces and f:U in S1 -> S2 is a differentiable mapping of an open set U, such that the differential of the map at p in U is an isomorphism, the f is a local diffeomorphism at p.

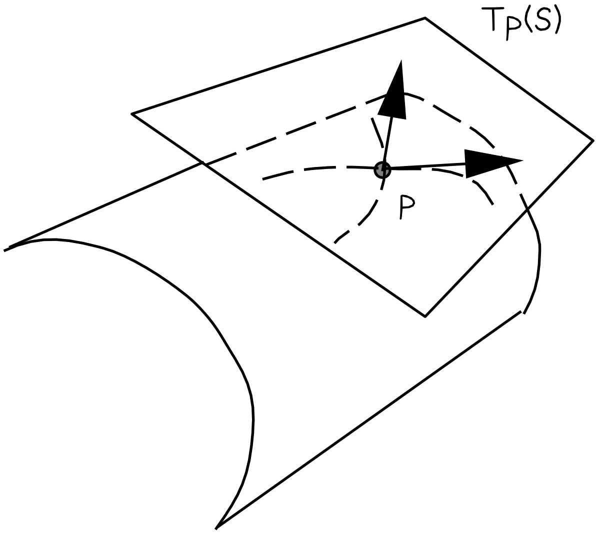
Geometry

A lot of geometry in R^n depends on the notion of inner product:

Angle between vectors
Orthogonality
The length of a vector,
Area

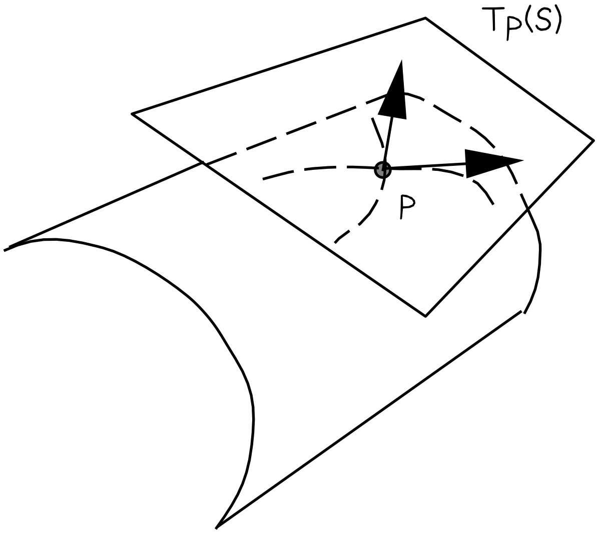


We want to generalize these ideas to a surface.



First Fundamental Form

Note: The inner product in R^3 induces an inner product on the vectors in any tangent space $T_p(S)$.



<u>Defn:</u> Let $\langle .,. \rangle_p$: $T_p(S) \times T_p(S) \to R$ denote the inner product induced on vectors of $T_p(S)$ by the inner product in the ambient R^3 space.

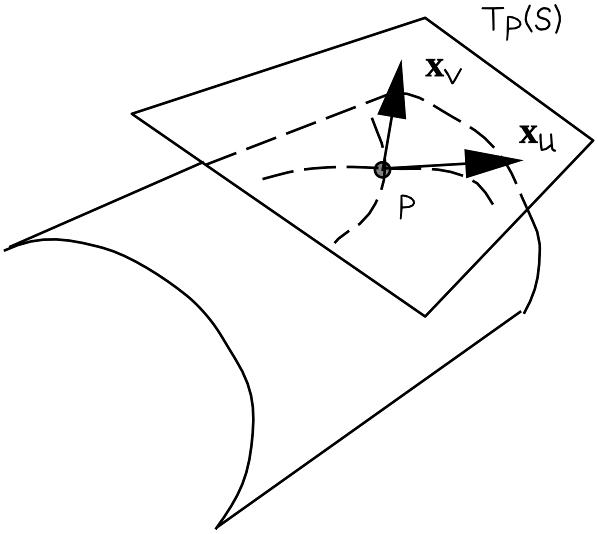
<u>Defn</u>: The function $I_p(w)$: $T_p(S) \to R$ defined by

$I_p(w) = \langle w,w \rangle = |w|^2 > 0$ is called the first fundamental form of the regular surface.

Note: The fundamental form is a function. It is a quadratic form.

First Fundamental Form

<u>Calculation;</u>



Recall: Any tangent vector w in $T_p(S)$ can be written as

$$w = u\,\mathbf{x}_u + v\,\mathbf{x}_v.$$

Thus, $I_p(w) = \langle u\,\mathbf{x}_u + v\,\mathbf{x}_v, u\,\mathbf{x}_u + v\,\mathbf{x}_v\rangle_p$

$$= \langle \mathbf{x}_u,\mathbf{x}_u\rangle_p\, u^2 + 2\langle \mathbf{x}_u,\mathbf{x}_v\rangle_p\, uv + \langle \mathbf{x}_v,\mathbf{x}_v\rangle_p\, v^2$$

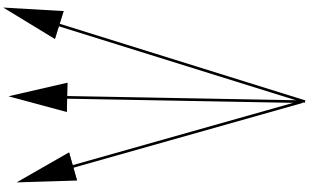
$$= Eu^2 + 2Fuv + Gv^2$$

where

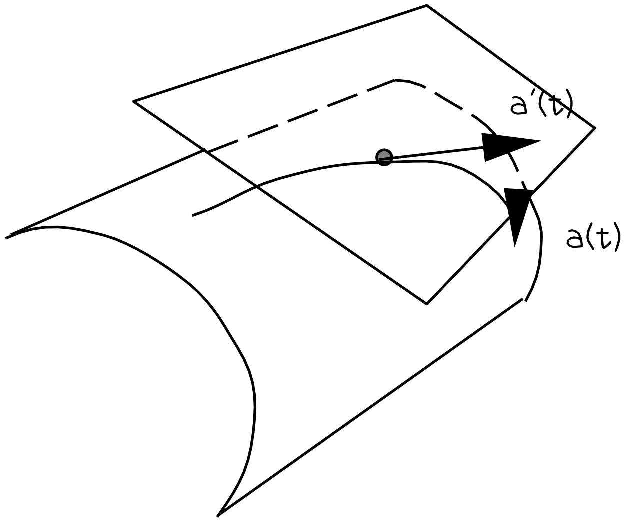
$$E(u,v) = \langle \mathbf{x}_u,\mathbf{x}_u\rangle_p$$

$$F(u,v) = \langle \mathbf{x}_u,\mathbf{x}_v\rangle_p$$

$$G(u,v) = \langle \mathbf{x}_v,\mathbf{x}_v\rangle_p.$$



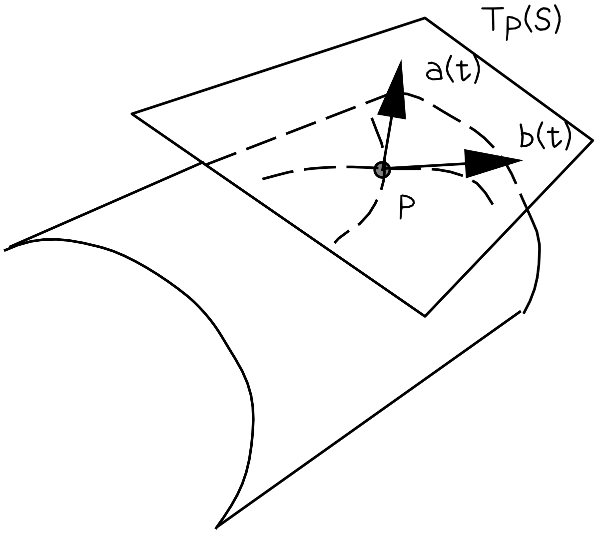
First Fundamental Form



Arc length: Let $a(t): I \to S$ be a curve on the surface S. Then, its incremental arclength ds at p is given by

$$ds = \sqrt{I_p(a'(t))}\ dt$$

First Fundamental Form

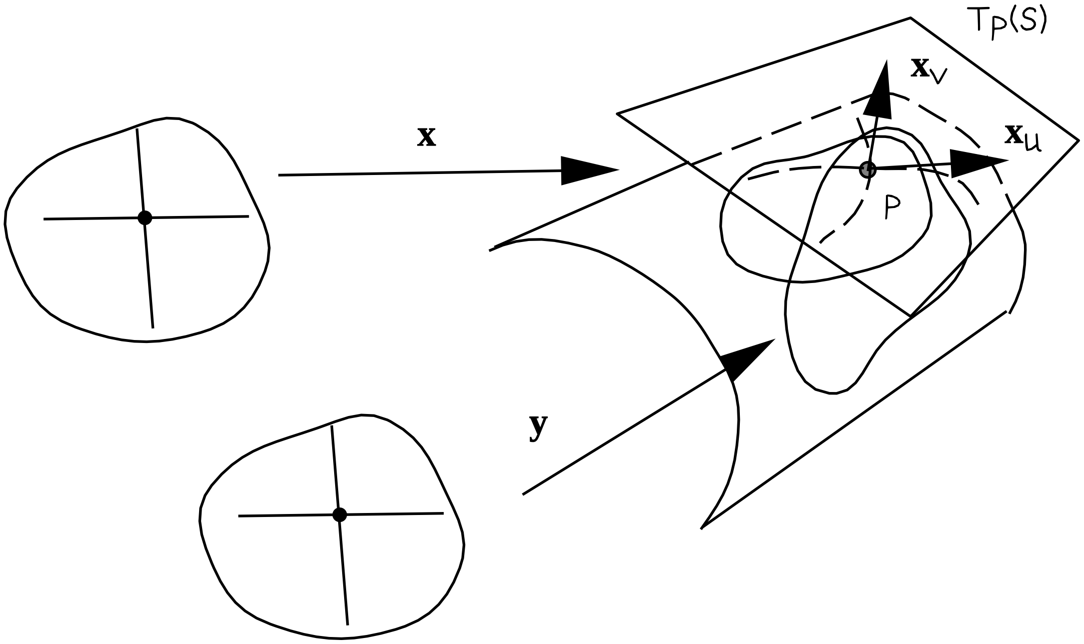


Angle: Let a(t):I->S and b(t):I->S be two curves on the surface intersecting at t=to at a point p on the surface. Then, if ϕ is the angle between them, then

$$\cos\phi = \frac{\langle a'(t), b'(t)\rangle_P}{|a'(t)||b'(t)|}.$$

In particular for coordinate curves $\cos\phi = F/\mathrm{sqrt}(EG)$.

Area



Let x:U -> S and y:W->S be two parameterizations such that x(U) and y(W) contain a bounded region R of S (which is contained in a ball). Let $Q = x^{-1}(R)$ and $P = y^{-1}(R)$.

Now $$\iint_P |y_a \wedge y_b|\, da\,db = \iint_P |x_u \wedge x_v|\ |d(u,v)/d(a,b)|\, da\, db = \underbrace{\iint_Q |x_u \wedge x_v|\, du\, dv}_{\text{Area of R}}.$$

$$\text{Area of R} = \iint_Q \sqrt{EG-F^2}\ du\, dv$$